Notes on:
Rational Forward Looking SIR§

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Abstract

§In the effort to be clear to a reader not used to economic models, we might end-up being pedantic. We apologize for this. Bisin: New York University, wp.nyu.edu/albertobisin/, alberto.bisin@nyu.edu. Moro: Vanderbilt University, andreamoro.net, andrea@andreamoro.net. We thank Pedro Sant’Anna and Giorgio Topa for their helpful comments on a preliminary draft of this paper; to Gianluca Violante for help with the calibration.
1 Introduction

2 Behavioral SIR

SIR. Consider a population of ex-ante identical (representative) agent. The population is distributed across the state space \( \{S, I, R\} \) representing, respectively, Susceptible, Infected, Recovered, agents. We normalize the state space so that \( S + I + R = 1 \) and let \( h_t = [I_t, R_t] \) denote the vector representing the distribution of the population at time \( t \). The dynamics of the distribution of the population - the SIR model - is as follows:\(^1\)

\[
I_{t+1} = I_t + \pi C_t (1 - R_t - I_t) I_t \\
R_{t+1} = R_t + \rho I_t
\]

where \( \pi \) is the probability that a Susceptible agent get infected upon contact with a Infected agent, \( C_t \) is the average number of contacts of Susceptible agents with other agents (they cannot distinguish the state of the agents he/she has contact with) in period \( t \). We assume \( \pi \) is a given epidemiological parameter. We also assume each Susceptible agent rationally chooses the number of contacts he/she has with other agents in period \( t \) - we discuss this choice problem next.

Rational choice and equilibrium. Infected and Recovered agents have no choice problem. Any Susceptible agent can instead choose the number of contacts he/she has in period \( t, c_t \), at any \( t \).\(^2\) Note that all Susceptible agents are identical and hence they will choose the same \( c_t \) and hence the average \( C_t \) will be equal to \( c_t \). Nonetheless, we assume each agent is "small," the effect of his/her choice on the average is - in the limit - null, and hence we assume each agent takes as given \( C_t \); that is, each agent thinks that \( C_t \) does not depend on his/her choice of \( c_t \). Formally, the agent (we drop Susceptible) chooses a sequence \( \{c_t\}_{t=1}^{\infty} \), given a sequence \( \{h_t\}_{t=1}^{\infty} \).

Rational forward looking choice, in the jargon of economics, implies that the agent can perfectly forecast the dynamics of \( h_t \).\(^3\) More precisely, we assume each agent i) perfectly anticipates the sequence of average contacts \( \{C_t\}_{t=1}^{\infty} \); and ii) knows that the dynamics of \( h_t \) is governed by equations (1-2) (and knows initial conditions and how to solve the difference equations). As a consequence, perfectly anticipating \( \{C_t\}_{t=1}^{\infty} \), the agent in fact perfectly anticipates \( \{h_t\}_{t=1}^{\infty} \).

An equilibrium (Nash equilibrium in the jargon) is then formally defined as follows: a sequence \( \{C_t\}_{t=1}^{\infty} \) such that, given \( \{C_t\}_{t=1}^{\infty} \) (hence given \( \{h_t\}_{t=1}^{\infty} \) all agents

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\(^1\)We write it the dynamics in discrete time, again the common use, because the choice problem we will study is more easily expressed in discrete time.

\(^2\)The distinction between lower and upper case \( c \) is crucial - please hold on.

\(^3\)The dynamics of \( h_t \) is deterministic in SIR. If the dynamics were stochastic, rational forward looking behavior would imply that the agent forecast the stochastic process.
choose \( \{c_t\}_{t=1}^{\infty} = \{C_t\}_{t=1}^{\infty} \). Mathematically, this is a fixed point problem in the space of sequences \( \{C_t\}_{t=1}^{\infty} \).

Each agent chooses \( \{c_t\}_{t=1}^{\infty} \), given \( \{C_t\}_{t=1}^{\infty} \) (hence given \( \{h_t\}_{t=1}^{\infty} \), via equations 1-2), to maximize his/her present discounted utility, which we now construct. We assume the agent has utility \( u(c_t) \) in period \( t \) from contacts \( c_t \) (social interactions); but only if he/she is not Infected: the utility of an Infected agent is zero. Each agent’s present discounted utility at \( t = 1 \), given \( \{h_t\}_{t=1}^{\infty} \) is:

\[
\sum_{t=1}^{\infty} \delta^t (1 - p(c_t, h_t)) u(c_t)
\]

where \( p(c_t, h_t) \) is the probability of being infected at \( t \), that is \( p(c_t, h_t) = \pi c_t I_t \); and \( \delta < 1 \) is the discount rate. Note that the probability that the agent is infected depends on \( c_t \) - his/her contact, not the average ones. Note also that the present discounted utility is well defined and deterministic: the agent needs to forecast the probability of being infected at any time \( t \), but those are given by his/her perfect forecast of \( \{C_t\}_{t=1}^{\infty} \) and his ability to solve for \( \{h_t\}_{t=1}^{\infty} \) via equations 1-2.

Now, economists have standard assumptions and functional forma for \( u(c_t) \) which satisfy a whole array of properties we think they should satisfy, from a choice-theoretical point of view, and which perform "well" in a large number of economic problems we use them in. There’s also a whole array of results about the dynamic optimization problem and the fixed point problem. This is less important at this state - it’s just math - nothing really conceptual. Typically, we would write the problem recursively - looking for a solution of the optimization problem of the form \( c(I, R) \), solving the functional fixed point problem in the definition of the Bellman equation, for given \( C(I, R) \):

\[
V(I, R; C(I, R)) = \max_c u(c) + \delta (1 - p(c, I)) V(I', R'; C(I', R')) ;
\]
\[
p(c, I) = \pi c I
\]
\[
I' = I + \pi C(I, R)(1 - R - I)I
\]
\[
R' = R + \rho I
\]

The first order conditions (necessary and sufficient under concavity) are:

\[
u'(c) = \delta \pi IV(I', R'; C(I', R'))
\]

The solution of this problem is a function \( c(I, R; C(I, R)) \). The equilibrium is a \( C(I, R) \) such that

\[
c(I, R; C(I, R) = C(I, R).
\]

\[\text{Initial conditions } I_0, R_0, \text{ are also given - we avoid notation for simplicity.}\]
We typically know existence of a solution and we know how to compute one.

A good part of the literature - also in economics - ha written down this kind of model but has made assumptions to avoid the fixed point complications. This is typically "not allowed" in economics, however. For instance, assume the agent thinks that his/her present value utility if he/she does not get infected, \( V(h') \) is a constant. This basically implies that the agent is not rational forward looking - that he/she believes that \( h \) is constant. Assume for simplicity that the utility function is logs, \( u(c) = \ln c \) (a typical work-horse in economics).

In this case, look at the First Order Condition 7,

\[
c = \frac{1}{\delta \pi I}.
\]

The SIR now has a straightforward form - going back to continuous time

\[
\dot{I} = \frac{1}{\delta}(1 - R - I) \tag{8}
\]
\[
\dot{R} = \rho I \tag{9}
\]

which can easily be solved in closed form.