1. Tautological Necessity aka “Tautology”

*Def:* A sentence S is a tautological necessity iff it is true in every row of its truth table.

Ex. 1

<table>
<thead>
<tr>
<th>Cube(a)</th>
<th>Cube(a) v ¬Cube(a)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>✓</td>
</tr>
<tr>
<td>F</td>
<td>✓</td>
</tr>
</tbody>
</table>

This truth table shows that Cube(a)v¬Cube(a) is a tautological necessity. There is no row in which the sentence is false. It is an instance of what’s sometimes called the law of the excluded middle. For any sentence A, either it is true or it is false.

Think of the rows of the truth table as corresponding to sets of Tarski’s World models, defined by the state of affairs represented in the reference rows. E.g., the first row of this truth table corresponds to the set of TW models in which a is a cube. (There are a lot.) The second corresponds to those in which a is not a cube. The second row in the truth table in Ex. 2 corresponds to the set of TW models in which a is a cube and b is not a cube.

Ex. 2

<table>
<thead>
<tr>
<th>Cube(a)</th>
<th>Cube(b)</th>
<th>Cube(a) v ¬Cube(b)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>✓</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>✓</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>✓</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>✓</td>
</tr>
</tbody>
</table>

Cube(a)v¬Cube(b) is not a tautological necessity. It is false in the third row of its truth table.

Ex. 3

<table>
<thead>
<tr>
<th>a = a</th>
<th>a = a</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

a=a is not a tautological necessity, either. It is false in the second row of its truth table.

But we know this sentence is a logical necessity. It can’t be false. What is going on?
The second row of the truth table represents a state of affairs that is not logically possible. It does not actually define a set of TW models, i.e., there is no TW model in which \( a \) isn’t the same as \( a \). More than that, there is no world, Tarski’s or otherwise, in which this is true. The truth table can still represent this state of affairs.

Consider that natural language can also represent this state of affairs. I just used it do so: \( a \) isn’t the same as \( a \). I can also say this: it’s false that \( a \) is the same as itself. We can formulate these sentences in English, knowing they do not represent a logically possible state of affairs. We don’t eliminate such sentences from the language, though. Similarly, we can construct the last row of this truth table, knowing it does not represent a logically possible state of affairs. We don’t eliminate such rows from our truth tables, though.

The definition of tautological necessity (the definition of each tautological concept) does not take any account of the possibility that some reference rows represent logically impossible states of affairs. The definition is simple. It could be followed by, for example, a computer that had no way of recognizing the meanings of atomic sentences. A sentence is not a tautological necessity if and only if it is false in any row of its truth table, even if that row does not represent a logically possible state of affairs.

A sentence that is not a tautological necessity might still be a logical necessity. \( a=a \) is an example. All tautological necessities are logical necessities, but not all logical necessities are tautological necessities.

2. Tautological Equivalence

Def: Two sentences \( S_1 \) and \( S_2 \) are tautologically equivalent iff they have all the same truth values in their joint truth table.

Ex. 1

\[
\begin{array}{c|c|c|c|c}
(1) & (2) & \neg(A \land B) & \neg A \lor \neg B \\
T & T & F & T \\
T & F & F & F \\
F & T & F & T \\
F & F & F & F \\
\end{array}
\]

These sentences are tautologically equivalent. Both have the same truth values in every row of their joint truth table. (They are an instance of a DeMorgan’s equivalence.)

Ex. 2

\[
\begin{array}{c|c|c|c|c}
(1) & (2) & \neg a \lor \neg b & \neg b \lor \neg a \\
T & T & F & F \\
T & F & F & F \\
F & T & F & F \\
F & F & F & F \\
\end{array}
\]
These two sentences are not tautologically equivalent. They have different truth values in the second row and in third row of their joint truth table. (Either row on its own is enough to conclude that they are not tautologically equivalent.)

The sentences are logically equivalent, however. The second and third rows represent logically impossible states of affairs. Again, tautological concepts do not take this into account.

3. Tautological Consequence

*Def:* Some sentence $S$ is a tautological consequence of some premises $P_1,\ldots, P_n$ iff there is no row in their joint truth table in which all of $P_1,\ldots, P_n$ are true and $S$ is false.

**Ex. 1**

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

This truth table shows more than one thing about tautological consequence. (The truth table has no way of sorting sentences into premises and conclusion. You need to read off the information you want.) It shows that $\neg A \lor \neg B$ is a tautological consequence of $\neg A$. There are no rows in which $\neg A$ is true and $\neg A \lor \neg B$ is false. It also shows that the reverse is not true. $\neg A$ is not a tautological consequence of $\neg A \lor \neg B$. In the second row, the latter is true and the former is false.

**Ex. 2**

Typically, we are curious about a particular argument, whether it is a case of a tautological consequence. What about the following argument?

$$
\begin{align*}
\neg A \\
\hline
\neg A \land \neg B
\end{align*}
$$

Is this a case of tautological consequence? Consider the joint truth table:

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
</tr>
</tbody>
</table>
It shows that \( \neg A \land \neg B \) is not a tautological consequence of \( \neg A \). The latter true and the former is false in the third row.

Ex. 3

Consider the following argument:

\[
\begin{array}{c|c|c}
\text{Cube(a)} & \text{a=b} & \text{Cube(b)} \\
\hline
T & T & T \\
T & T & F \\
T & F & T \\
T & F & F \\
F & T & T \\
F & T & F \\
F & F & T \\
F & F & F \\
\end{array}
\]

We can recognize this as a case of logical consequence. It is impossible for the premises to be true and the conclusion to be false. We could also give a formal proof of it. Is it a case of tautological consequence, though? Here is the joint truth table:

\[
\begin{array}{c|c|c|c|c|c|c|c}
\text{(1)} & \text{(2)} & \text{(3)} & \text{(1)} & \text{(2)} & \text{(3)} \\
\hline
\text{Cube(a)} & \text{a=b} & \text{Cube(b)} & \text{Cube(a)} & \text{a=b} & \text{Cube(b)} \\
\hline
T & T & T & T & T & T \\
T & T & F & T & T & F \\
T & F & T & T & T & F \\
T & F & F & F & F & T \\
F & T & T & F & T & T \\
F & T & F & F & T & F \\
F & F & T & T & F & T \\
F & F & F & F & F & F \\
\end{array}
\]

It shows that the argument is not a case of tautological consequence. In the second row, the premises are true and the conclusion is false. As in the analogous cases above, it does not matter that the second reference row represents a logically impossible state of affairs. Again, natural language sometimes does the very same thing. E.g., it is true that \( a \) is a cube and that it’s the same thing as \( b \), but \( b \) isn’t a cube.