

Financial Contagion Revisited

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1 Introduction

The financial crisis of 2007-9 has demonstrated the importance of dislocation in the financial sector as a cause of economic fluctuations. The prevalence of financial crises has led many to conclude that the financial sector is unusually susceptible to shocks. One theory is that small shocks, which initially only affect a few institutions or a particular region of the economy, spread by *contagion* to the rest of the financial sector and then infect the larger economy. There is a growing literature on this phenomenon. Excellent surveys are provided by Glasserman and Young (2016) and Benoit et al. (2017). In this paper we focus on contagion through interbank markets. Allen and Gale (2000) developed a stylized model with simple networks of four banks to consider the trade-off between risk sharing through interbank markets and the possibility of contagion originating from small shocks. They show that complete networks where every bank is connected with every other bank are more robust than incomplete markets where not all banks are connected. Recent literature has focused on more general models and a range of sizes of shocks. Elliott, Golub and Jackson (2014) consider the role of integration and diversification. The former is concerned with how much banks rely on other banks while the latter is the number of banks a particular bank's liabilities are spread over. In the context of different models, Gai and Kapadia (2010) and Acemoglu, Ozdaglar, and Tahbaz-Salehi (2015) show that connected networks where all banks are connected to each other, at least indirectly, are more robust to shocks because of risk sharing than networks where they are not all connected. However, large shock are more likely to make all institutions fail in in connected networks.

This work is related to Joseph Stiglitz's career-long interest in finance and banking (see, e.g., Stiglitz (1969), (1972a,b), Cass and Stiglitz (1972), Stiglitz (1973), Grossman and Stiglitz (1980), Stiglitz and Weiss (1981), and Hellmann, Murdoch and Stiglitz (2000)). In particular, it is related to his recent work on contagion and financial networks (Battiston et al. (2007, 2012a, b) and Battiston et al. (2016)).

In this paper, we again focus on the role of small shocks in bringing down the financial system through interbank market connections. We are able to show that general complete networks are more robust than incomplete networks. As before, we take as our starting point the model presented in Allen and Gale (1998). The assumptions about technology and preferences have become standard in the literature since the appearance of the Diamond

and Dybvig (1983) model. There are three dates $t = 0, 1, 2$ and at the first date there is a large number of identical consumers, each of whom is endowed with one unit of a homogeneous consumption good. At the beginning of the second date, the consumers learn whether they are early consumers, who only value consumption at date 1, or late consumers, who only value consumption at date 2. Uncertainty about their preferences creates a demand for liquidity. In order to provide for future consumption, consumers have to save their endowment. Two assets are available for this purpose, a safe, short-term asset and a risky, long-term asset. We refer to these as the short and long assets, respectively. Uncertainty about their preferences creates a demand for liquidity. The long asset has a higher return, but it pays off only in the last period and is therefore not useful for providing consumption to early consumers.

Banks have a comparative advantage in providing liquidity. At the first date, consumers deposit their endowments in the banks, which invest them in the long and short asset. In exchange, depositors are promised a fixed amount of consumption at each subsequent date, depending on when they choose to withdraw. Early consumers withdraw at the second date while late consumers withdraw at the third date. The banking sector is perfectly competitive, so banks offer risk-sharing contracts that maximize depositors' ex ante expected utility, subject to a zero-profit constraint.

Bank runs occur in some states because the returns to the risky asset are so low that the banks cannot make the payments promised to their depositors. Depositors also have access to the short asset. If late consumers anticipate that their payout at date 2 will be less than depositors will receive at date 1 they will withdraw early, pretending to be early consumers, and save the proceeds using the short asset until the final date.

In Allen and Gale (1998), the returns to the risky asset are perfectly correlated across banks, so a low return causes insolvency in all banks simultaneously. A bad shock is thus tantamount to an economy-wide financial crisis. In the present paper, by contrast, we are explicitly interested in constructing a model in which small shocks lead to large effects by means of contagion. The question we address is whether and under what circumstances a shock within a single (small) sector can spread to other sectors and lead to an economy-wide financial crisis by contagion.

The economy consists of a number of sectors or regions. For simplicity, it is assumed that the long asset can be liquidated at date 1 or at date 2. The risk-free returns are $r < 1$ and $R > 1$, respectively. The number of early

and late consumers in each region is assumed to be random. These liquidity shocks are imperfectly correlated across regions, thus providing the potential for insurance against the liquidity shocks. Regions with high liquidity shocks can obtain liquidity from low consumers. One way to provide this liquidity insurance is by exchanging deposits. Suppose that region A has a large number of early consumers when region B has a low number of early consumers, and vice versa. Since regions A and B are otherwise identical, their deposits are perfect substitutes and the banks can exchange deposits at the first date without affecting their net wealth. After the liquidity shocks are observed, one region will have a high demand for liquidity and one will have a low demand. Suppose region A has a higher than average number of early consumers. Then banks in region A can meet their obligations by liquidating some of their deposits in the banks of region B . Region B is happy to oblige, because it has an excess supply of liquidity in the form of the short asset. Later, banks in region B will want to liquidate the deposits they hold in the banks of region A to meet the above-average demand from late consumers in region B . The banks in region A can meet this demand because they have a below-average number of late consumers, that is, an excess supply of the long asset.

In general, whenever the liquidity shocks in different regions are less than perfectly correlated, banks can improve the risk sharing they offer to depositors through cross holdings of deposits. In certain circumstances, it can be shown that complete risk sharing can be achieved in this way: as long as all the regions are connected in a particular way and there is no aggregate uncertainty across all regions, then the first-best allocation can be decentralized through a competitive banking sector.

Inter-regional cross holdings have another role: they create an interdependency among the regions that is one of the ingredients needed for financial contagion. Financial contagion is a complicated phenomenon and it requires several pre-conditions. The first one has already been mentioned, the financial interconnectedness that arises from cross holdings of deposits or other financial claims. The second element is that there must be an aggregate shortage of liquidity. To understand what this means, we need to be more precise about the conditions of individual banks. We distinguish three conditions in which banks can find themselves at the second date, after uncertainty has been resolved. A bank is solvent if the demand for withdrawals is less than the value of liquid assets (the short asset plus net holdings of deposits in other banks). A bank is insolvent if it can meet the demand for withdrawals,

but only by liquidating the long asset at date 1. It can reduce the payout to late consumers as long as it pays them as much as the early consumers (otherwise there will be a run). The value of the long asset that can be liquidated at the second date, consistently with the incentive constraint, is called the bank's buffer. This buffer, added to the value of liquid assets, is the maximum that the bank can provide in the second period, without provoking a run. Bankruptcy occurs when the demand for withdrawals is greater than the sum of the liquid assets and the buffer. Then the late consumers run and all the creditors cash in their claims. The bank is forced to liquidate all its assets and still cannot meet its promised payments.

Banks try to avoid insolvency, because liquidating the long asset reduces the value of the bank. Instead of getting a return of R per unit at date 2, they get a return of r per unit at date 1. As long as the bank has more of the short asset than it needs, it is happy to redeem its deposits by paying out the short asset. When the bank does not have an excess supply of the short asset, it tries to meet any shortfall by liquidating deposits in other banks. The problem arises when there is a global excess demand for the short asset. Cross holdings of deposits can be used to redistribute excess supplies of the short asset among regions but they cannot they cannot increase the total amount of the short asset in existence at date 1. So, when there is an economy-wide excess demand for the short asset, it can only be met by liquidating the long asset, but banks will only do this if they are forced to do so, that is, if the demand for withdrawals is greater than the liquid assets the bank holds.

Suppose that the global excess demand for liquidity is attributable to an extremely high demand for liquidity (number of early consumers) in one region. While the excess demand for liquidity may be small by comparison to the entire economy, it may be very large in relation to the region's assets, large enough to cause bankruptcy in that region. If the banks in this region were able to call upon the other regions, by withdrawing the deposits held in those regions, then it could possibly avoid bankruptcy. But the banks in other regions do not want to provide liquidity and they can avoid doing so by using their extra-regional deposits strategically. For example, suppose that region A has a high demand for liquidity and region B has an average demand for liquidity. Region B has just enough of the short asset to meet the demands of its own depositors, without giving anything to banks in region A . The two regions have the same number of extra regional deposits. If region A tries to liquidate its deposits in region B to get more liquidity, then region

B will avoid liquidating the long asset by liquidating its claims on banks in region A instead. These two transactions cancel out, leaving region A no better off. Even if there are many regions and the pattern of cross holdings is complicated, the same principle applies. As long as region A is merely insolvent, it cannot force the other regions to provide liquidity.

Things are different once banks in region A become bankrupt, because then a deposit in region A is worth less than a deposit in region B . If the banks in various regions simultaneously try to liquidate their cross holdings, there will be a transfer of value to the bankrupt regions. This spillover is what allows for the possibility of contagion, but whether contagion occurs or not depends crucially on the form of connectedness between regions as well as the other parameters of the model.

If every region is connected to every other region, then there may be no contagion at all. Suppose that markets are complete, in the sense that a bank in one region can hold deposits in all other regions. Then bankruptcy in one region will put pressure on all of the banks. However, if there are many regions, so that the number of bankrupt banks is small relative to the total number of banks, the transfer that each region has to make will be small and that region's buffer will be big enough to cover the transfer demanded. As a result, there may be insolvency but no bankruptcy outside the troubled region.

The impact of insolvency in one region is quite different if markets are incomplete, in the sense that the banks in one region are able to hold deposits in only a few of the other regions. In this case, the transfer occasioned by bank runs in one region fall initially on a few regions outside the initially troubled region. The total size of the buffer held by these regions may not be enough to sustain the demands made on it and those regions directly connected to the initially troubled region may be driven into bankruptcy. Once this happens, it is easier for the contagion to spread. There is now a group of bankrupt regions, in which the run by depositors has forced the banks to liquidate all their assets, with a consequent loss in value. Furthermore, their claims on the remaining solvent regions may be smaller in proportion to their liabilities than in the original region where the contagion began. This is because their claims on each other are now of less value. And so it may be easier, under certain conditions, for the contagion to spread. We provide conditions under which insolvency will spread to all the regions by contagion when markets are incomplete, and there would be no contagion if markets were complete.

It is important to note the importance of the free rider problem in explaining the difference between complete and incomplete markets. With complete markets, every bank in every region suffers a loss from the troubled region. There is no way to avoid paying one's share. With incomplete markets, the banks in the troubled region have a direct claim only on the banks in a small number of regions. The banks in those regions have claims on banks in other regions and indirectly on all the regions. But as long as the regions are solvent, they can decline to offer liquidity if it means liquidating the long asset. They do this by liquidating their deposits in other regions instead. The effect of this policy is to make things worse for the regions that are directly connected to the troubled region, to the point where they too become insolvent. At that point, some of the regions that refused to provide liquidity find themselves on the front line, holding claims on insolvent regions. The attempt to protect oneself by hoarding liquidity and refusing to liquidate costly assets proves ultimately self-defeating and makes the situation worse.

The rest of the paper is organized as follows.

2 Liquidity Preference

In this section we describe a simple model in which liquidity preference leads to a demand for risk-sharing contracts. The framework borrows from the models in Diamond and Dybvig (1983) and Allen and Gale (1998), with some significant differences.

Dates. There are three dates $t = 0, 1, 2$.

Goods. At each date there is a single consumption good, which serves as a numeraire. This good can also be invested to provide for future consumption. Each consumer is endowed with one unit of the good at date 0 and nothing at the subsequent dates.

Assets. There are two types of assets, a liquid asset (the *short asset*) and an illiquid asset (the *long asset*).

- – The short asset is represented by a storage technology. An investment of one unit of the good at date $t = 0, 1$ yields one unit of the good at date $t + 1$.

- Investment in the long asset can only be made at date 0. One unit of the good invested in the long asset at date 0 produces R units of the good at date 2. The long asset can be prematurely liquidated at date 1, in which case it yields a liquidation value of r per unit. We assume that $0 < r < 1 < R$. The decision whether to liquidate the asset at date 1 or to let it mature at date 2 is made at the beginning of date 1.

Regions. There are n (ex ante) identical regions $i = 1, \dots, n$. In each region there is a continuum of (ex ante) identical consumers. Let $0 < \omega^i < 1$ be the random fraction of early consumers in region i and $1 - \omega^i$ be the random fraction of late consumers.

Uncertainty. The (finite) set of states of nature is denoted by Ω with generic element $\omega = (\omega^1, \dots, \omega^n)$ and probability density $p(\omega) > 0$. The random variables $\{\omega^i\}$ are *exchangeable*: for any permutation $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ and any state $\omega = (\omega^1, \dots, \omega^n)$, the state ω' defined by putting

$$\omega' = (\omega^{\pi(1)}, \dots, \omega^{\pi(n)})$$

also belongs Ω and satisfies $p(\omega) = p(\omega')$. In particular, this implies that the random variables $\{\omega^i\}$ have the same marginal distributions.

Information. All uncertainty is resolved at date 1 when each consumer observes the state of nature ω and learns whether he is an early or late consumer. A consumer's type is not observable, so late consumers can always imitate early consumers.

Preferences. A typical consumer's utility function in region i can be written as

$$U^i(c_1, c_2) = \begin{cases} u(c_1) & \text{with probability } \omega^i \\ u(c_2) & \text{with probability } 1 - \omega^i, \end{cases} \quad (1)$$

where c_t denotes consumption at date $t = 1, 2$. The period utility functions $u(\cdot)$ are assumed to be twice continuously differentiable, increasing and strictly concave.

Since ω^i is also the probability of being an early consumer at date 1, the welfare of a consumer at date 0 is given by the expected utility

$$\sum_{\omega \in \Omega} p(\omega) \{ \omega^i u(c_1(\omega)) + (1 - \omega^i) u(c_2(\omega)) \}.$$

The role of banks is to make investments on behalf of consumers. We assume that only banks can distinguish the genuine long assets from assets that have no value. Any consumer who tries to purchase the long asset faces an extreme adverse selection problem, so in practice only banks will hold the long asset. This gives the bank an advantage over consumers in two respects. First, the banks can hold a portfolio consisting of both types of assets, which will typically be preferred to a portfolio consisting of the short asset alone. Secondly, by pooling the assets of a large number of consumers, the bank can offer insurance to consumers against their uncertain liquidity demands, giving the early consumers some of the benefits of the high-yielding risky asset without subjecting them to the volatility of the asset market.

Free entry into the banking industry forces banks to compete by offering deposit contracts that maximize the expected utility of the consumers. Thus, the behavior of the banking industry in each region can be represented by an optimal risk-sharing problem. The behavior of banks will be discussed in more detail later, when we describe an equilibrium with decentralized banking. First, however, we look at a benchmark case where a central planner makes optimal decisions on behalf of the consumers in all regions.

3 Optimal Risk-Sharing

The planner is assumed to maximize the sum of consumer's expected utilities. The planner holds a portfolio (x_t, y_t) at the end of date $t = 0, 1$, where x_t is the total amount of the long asset and y_t is the total amount of the short asset. Consumption in region i at date $t = 1, 2$ is denoted by $c_t^i(\omega)$ and depends on the state, which is revealed at the beginning of date 1. The general problem can be written as follows:

$$(P1) \quad \left\{ \begin{array}{l} \max \quad \sum_{i=1}^n \sum_{\omega \in \Omega} p(\omega) \{ \omega^i u(c_1^i(\omega)) + (1 - \omega^i) u(c_2^i(\omega)) \} \\ \text{s.t.} \quad \text{(i)} \quad x_0 + y_0 \leq n; \\ \quad \quad \text{(ii)} \quad \sum_{i=1}^n \omega^i c_1^i(\omega) \leq y_0 - y_1(\omega) + r(x_0 - x_1(\omega)); \\ \quad \quad \text{(iii)} \quad \sum_{i=1}^n (1 - \omega^i) c_2^i(\omega) \leq y_1(\omega) + R x_1(\omega); \\ \quad \quad \text{(iv)} \quad c_1^i(\omega) \leq c_2^i(\omega). \end{array} \right.$$

The objective function is the sum of expected utilities. Conditions (i), (ii) and (iii) are the budget constraints at dates 0, 1 and 2, respectively. The last constraint, condition (iv), is the incentive constraint, which says that late

consumers do not wish to imitate early consumers in any region and in any state.

We begin by studying a modified problem in which the incentive constraint (iv) is omitted. The set of attainable consumption allocations $\{(c_1^i, c_2^i)\}$ in the modified problem is convex and the utility function $u(\cdot)$ is concave. It follows from general risk-sharing principles that consumption will be uniform across regions and that aggregate consumption depends only on the aggregate shock $\hat{\omega} \equiv \sum_{i=1}^n \omega^i$. Then we can write the planner's problem in a simpler form, treating consumption $(c_1(\hat{\omega}), c_2(\hat{\omega}))$ as a function of $\hat{\omega}$ and the date only. With a slight abuse of notation we write $p(\hat{\omega})$ for the probability of $\hat{\omega}$. Then the planner has to choose a total investment x_0 in the long asset, a total investment y_0 in the short asset, an amount of the long asset $x_1(\hat{\omega})$ to carry through to date 2, an amount $y_1(\hat{\omega})$ of the short asset to carry through to date 2, the consumption $c_1(\hat{\omega})$ of an early consumer, and the consumption $c_2(\hat{\omega})$ of a late consumer in order to maximize the typical consumer's expected utility. Note that the initial investment portfolio (x_0, y_0) does not depend on $\hat{\omega}$ because the planner does not yet know the value of $\hat{\omega}$. However, all the decisions made at date 1 and date 2 depend on $\hat{\omega}$, which is revealed at the beginning of date 1.

The modified risk-sharing problem can be written in per capita terms as follows:

$$(P2) \quad \begin{cases} \max & \sum_{\hat{\omega}} p(\hat{\omega}) \{ \hat{\omega} u(c_1(\hat{\omega})) + (1 - \hat{\omega}) u(c_2(\hat{\omega})) \} \\ \text{s.t.} & \text{(i)} \quad x_0 + y_0 \leq n; \\ & \text{(ii)} \quad \hat{\omega} c_1(\hat{\omega}) \leq y_0 - y_1(\hat{\omega}) + r(x_0 - x_1(\hat{\omega})); \\ & \text{(iii)} \quad (1 - \hat{\omega}) c_2(\hat{\omega}) \leq y_1(\hat{\omega}) + R x_1(\hat{\omega}). \end{cases}$$

This problem is easy to solve and it turns out that it satisfies the first-order condition

$$u'(c_1(\hat{\omega})) \geq u'(c_2(\hat{\omega})).$$

If this condition were not satisfied for some aggregate shock $\hat{\omega}$, the objective function could be increased by using the short asset to shift some consumption from early to late consumers. Thus, a solution to (P2) satisfies the incentive constraints of (P1) and hence must be a solution to (P1). In particular, this means that a solution to (P1) achieves the first best because the incentive constraints are non-binding.

Theorem 1 *The planner's risk-sharing problem (P1) is equivalent to the modified problem (P2). From this it follows that the solution to the planner's problem is first-best efficient, that is, the incentive constraints do not bind.*

It is worth noting the special case in which there is no aggregate uncertainty, that is, $\hat{\omega}$ is a constant across all states of nature. In that case, the optimum consumption profile (c_1, c_2) is non-stochastic. This case is of particular interest in the sequel because, when there is no aggregate uncertainty, the first best can be decentralized using standard (non-contingent) deposit contracts.

4 Equilibrium

In this section we describe the working of a decentralized banking system. But first we need to specify the banks' investments.

The central planner in Section 3 can insure consumers against liquidity shocks by re-allocating goods across regions. Unlike the central planner, the banks cannot directly allocate goods across regions. Instead they must operate through an *interbank market* in financial claims. The kinds of claims allowed in this model are deposits, that is, banks can trade bank deposits in different regions in order to provide insurance against liquidity shocks. Since deposits are homogenous within a region, we can assume that a bank in region i is only interested in holding deposits in a representative bank in any other region j . To allow for the possibility that markets are not complete, we introduce the notion of a market structure.

Market structure: For any region i there is a set of neighboring or adjacent regions $N^i \subset \{1, \dots, i-1, i+1, \dots, n\}$. A bank in region i is allowed to hold deposits in a representative bank in the regions $j \in N^i$ and is not allowed to hold deposits in banks in regions $j \notin N^i$. Consumers can only hold deposits in a bank in their own region.

The interbank deposit markets is said to be *complete* if banks are allowed to hold deposits in all other regions, that is,

$$N^i = \{1, \dots, i-1, i+1, \dots, n\}$$

for each region i . Otherwise the interbank market is said to be *incomplete*. Whether the interbank market is complete or incomplete, we always assume

that all the regions are connected in an intuitive sense. Region i is *directly connected* to region j if $j \in N^i$. Region i is *indirectly connected* to region j if there exists a sequence $\{i_1, \dots, i_K\}$ such that $i_1 = i$, $i_K = j$, and i_k is directly connected to region i_{k+1} for $k = 1, \dots, K - 1$. Finally, the deposit market is said to be *connected* if, for every ordered pair of regions (i, j) with $i \neq j$, region i is (indirectly) connected to region j .

In each region i there is a continuum of identical banks. All banks in a given region are assumed to behave in the same way and all consumers in a given region are assumed to behave in the same way. Thus, we can describe an equilibrium in terms of the behavior of a representative bank and representative consumer (or one early and one late consumer at dates $t = 1, 2$) in each region.

At the first date, consumers in region i can deposit their endowment of the consumption good with a bank in exchange for a *deposit contract* that promises them either c_1^i units of the good at date 1 or c_2^i units of the good at date 2. The bank will not necessarily keep this promise and a lot of attention will be paid in what follows to the rules governing the bank when it is unable to meet its commitments.

Each bank in region i takes the resources deposited by the consumers and invests them in a portfolio (x_0^i, y_0^i, z_0^i) consisting of $x_0^i \geq 0$ units of the short asset, $y_0^i \geq 0$ units of the short asset and an admissible portfolio of deposits $z_0^i \geq 0$ held in other regions. A deposit portfolio for region i is an $(n - 1)$ -tuple $z_0^i = (z_1^i, \dots, z_{i-1}^i, z_{i+1}^i, \dots, z_n^i)$, where z_{0j}^i is the number of deposits in region j held by the bank in region i . The portfolio z_0^i is admissible if $z_{0j}^i = 0$ for any region j that is not adjacent to i , that is, $j \notin N^i$. Let Z^i denote the admissible set of portfolios for region i .

At the beginning of the second period the state of nature ω is observed and individual consumers learn whether they are early consumers or late consumers. Late consumers can calculate whether they are better off withdrawing their deposits immediately or waiting to withdraw their deposits in the last period. If it is weakly optimal to wait we assume that they do so in equilibrium; otherwise they withdraw immediately. In the latter case, there is a run and the bank is forced to liquidate all its assets in order to meet the demands of the depositors. The banks choose a portfolio $(x_1^i(\omega), y_1^i(\omega), z_1^i(\omega))$ of assets to carry forward into the next period.

In the final period, there are no decisions to make. The banks liquidate all their assets and distribute the proceeds to the depositors who consume them.

The equilibrium is defined recursively, beginning with the last period.

4.1 Equilibrium in the Final Period

In the final period, banks liquidate all of their assets and distribute the proceeds to their depositors. The typical bank in region i has a portfolio $(x_1^i(\omega), y_1^i(\omega), z_1^i(\omega))$ at the beginning of date 2 in state of nature ω . There are two cases to be considered, depending on whether or not the banks were bankrupt at date 2.

When banks go bankrupt, they have to liquidate all of their assets immediately. Thus, if the banks in region i were bankrupt at date 1 then

$$(x_1^i(\omega), y_1^i(\omega), z_1^i(\omega)) = 0.$$

The value of the banks' deposits in that case is $q_2^i(\omega) = 0$.

In the other case, the banks in region i are not bankrupt at date 1, so they have assets to dispose of at date 2. Since these assets include deposits in other regions' banks, we have to take account of the value of deposits in other regions when calculating the value of deposits in region i . In other words, we have to determine the value of deposits in all regions simultaneously.

Without loss of generality, we can assume that all banks are bankrupt at date 2. At date 0, each bank will choose a value of c_2^i high enough so that

$$q_2^i(\omega) \leq c_2^i$$

in every state ω . This is a zero-profit condition resulting from perfect competition among the banks in each region. If it were violated, it would mean that some assets were left over in some state of nature and that would violate the assumption that banks choose deposit contracts to maximize consumer welfare.

The assets of the typical bank in region i are valued at

$$\sum_{j \neq i} q_2^j(\omega) z_{j1}^i(\omega) + y_1^i(\omega) + R x_1^i(\omega),$$

which is equal to the bank's claims on banks in regions $i-1$ and $i+1$ *plus* the bank's holding of the short asset *plus* the bank's holding of the long asset. The bank's liabilities are the number of deposits outstanding, each valued at

the market price $q_2^i(\omega)$

$$\left(\sum_{j \neq i} z_{i1}^j(\omega) + (1 - \omega^i) \right) q_2^i(\omega).$$

The equilibrium value of $q_2^i(\omega)$ is determined by the condition that the bank's assets just equal its liabilities, thus,

$$\left(\sum_{j \neq i} z_{i1}^j(\omega) + (1 - \omega^i) \right) q_2^i(\omega) = \sum_{j \neq i} q_2^j(\omega) z_{j1}^i(\omega) + y_1^i(\omega) + Rx_1^i(\omega). \quad (2)$$

The equilibrium values of the deposits are determined simultaneously by the n equations (2), one for each region, involving n unknowns, one price $q_2^i(\omega)$ for each region.

[It is easy to see that there is a solution to this system of equations for every non-negative specification of portfolios. For regions in which banks were insolvent at date 1 put $q_2^i(\omega) = 0$. For regions in which banks were solvent at date 1 we can obtain a lower bound by assuming $q_2^j(\omega) = 0$ on the right hand side of (2) and solving for $q_2^i(\omega)$. Next substitute these values into the right hand side of (2) and continue to perform this algorithm indefinitely. The values of $q_2^i(\omega)$ obtained at each iteration are non-decreasing. They are also bounded above, because denoting the left and right hand values by $q_2^i(\omega)$ and $q_2(\omega)$ respectively and summing (2) over i yields

$$\begin{aligned} \sum_{i=1}^n (1 - \omega^i) q_2^i(\omega) &= \sum_{i=1}^n \left(\sum_{j \neq i} q_2^j(\omega) z_{j1}^i(\omega) - \sum_{j \neq i} q_2^i(\omega) z_{i1}^j(\omega) \right) \\ &\quad + \sum_{i=1}^n (y_1^i(\omega) + Rx_1^i(\omega)) \\ &\leq \sum_{i=1}^n (y_1^i(\omega) + Rx_1^i(\omega)), \end{aligned}$$

since $q_2^i(\omega) \geq q_2(\omega)$. Thus, convergence is assured and by continuity the limiting values will satisfy (2) for $i = 1, \dots, n$.]

4.2 Equilibrium in the Intermediate Period

At the beginning of date 1 the banks in region i have the portfolio (x_0^i, y_0^i, z_0^i) chosen at date 0. These are their assets. The liabilities of the bank are the

potential claims from depositors. The total deposits outstanding from depositors in region i are one unit and from banks in other regions are $\sum_{j \neq i} z_{i0}^j$. The bank has promised each depositor c_1^i units of consumption on demand, but not all depositors will demand this payment in period 1. Early consumers have no choice but to withdraw their deposits at date 1. Late consumers can withdraw at date 2 or withdraw at date 1 and store the goods until date 2. It is optimal for the late consumers to wait and withdraw at the final date if and only if

$$q_2^i(\omega) \geq q_1^i(\omega).$$

Otherwise they would be better off withdrawing at date 1 and storing the deposit until date 2.

The next task is to determine the conditions under which the bank will be able to meet the claims made by the depositors. Suppose that the bank is not bankrupt. The bank needs to choose a new portfolio $(x_1^i(\omega), y_1^i(\omega), z_1^i(\omega)) \geq 0$ that satisfies the following condition:

$$\sum_{j \neq i} q_1^i(\omega)(z_{i0}^j - z_{i1}^j(\omega)) - \sum_{j \neq i} q_1^j(\omega)(z_{j0}^i - z_{j1}^i(\omega)) + \omega^i q_1^i(\omega) \leq (y_0^i - y_1^i(\omega)) + r(x_0^i - x_1^i(\omega)). \quad (3)$$

The left hand side represents the value of the deposits being redeemed at date 1 (where it is implicitly assumed that $q_1^i(\omega) = c_1^i$ since the bank is non-bankrupt).

The set of portfolios that satisfy the constraint (3) is compact and convex. If it is non-empty, then the bank should choose the portfolio from the set of portfolios that maximize the value of the deposits at date 2 (the bank cannot pay out more than c_1^i at date 1 but has chosen c_2^i large enough that additional assets can always be distributed at date 2). Let $\zeta_1^i(a_0, q, \omega)$ denote the set of optimal portfolios at date 1. It depends on the action $a_0^i = (x_0^i, y_0^i, z_0^i, c_1^i, c_2^i)$ chosen at date 0, on the prices q , and on the state ω .

If the set of feasible portfolios at date 1 is empty, then we put $\zeta_1^i(a_0, q, \omega) = \{0\}$ and force the bank to liquidate at date 1. In this case, both the banks in other regions and the late consumers in region i will withdraw their deposits in the current period and the value of the deposits is determined by the equation

$$\left(\sum_{j \neq i} z_{i0}^j + 1 \right) q_{i1}(\omega) = \sum_{j \neq i} q_{j1}(\omega) z_{j0}^i + y_0^i + r x_0^i.$$

The left hand side is the number of deposits outstanding multiplied by the liquidation value; the right hand side is the liquidation value of the assets, including the value of deposits in banks in other regions.

Where late consumers are indifferent between withdrawing their deposits at date 1 and date 2, we assume that they withdraw at the final date. We make this assumption because we want to avoid runs if at all possible. Under this assumption, it turns out that there are only two possibilities: either $q_2^1(\omega) \geq q_1^i(\omega) = c_1^i$ or $0 = q_2^i(\omega) \leq q_1^i(\omega) < c_1^i$ and there is a run that forces the bank to liquidate all its assets at the first date. To see this, suppose that $q_1^i(\omega) = c_1^i$. Then if $q_2^i(\omega) < c_1^i$ it must be optimal for all late consumers to withdraw at the first period and get c_1^i . But since consumption can be stored from date 1 to date 2, there must exist a set of decisions by the bank that provide $q_2^i(\omega) \geq c_1^i$, that is $\zeta_1^i(a_0, q, \omega) \neq \{0\}$, contradicting the equilibrium conditions. The other case that needs to be considered is $q_1^i(\omega) < c_1^i$. In that case, the bank is unable to meet its obligations and must liquidate all assets. This implies that $q_2^i(\omega) = 0$ as required.

4.3 Equilibrium in the Initial Period

At the first date, banks choose investment portfolios and deposit contracts to maximize the expected utility of the typical depositor, taking as given the behavior of the banks in other regions. So far we have described the equilibrium behavior of the banks at dates 1 and 2, assuming that all the banks in region i behave identically. But in order to describe the choice of the bank's optimal portfolio at the first date, we have to allow for the possibility that the bank chooses a different portfolio from the others and consequently will have a different liquidation value. In other words, we cannot take the liquidation value $q_t^i(\omega)$ as given when the bank is changing its initial portfolio. Similarly, we cannot assume that the bank can sell deposits to other banks; if the portfolio chosen by the bank is unattractive to other banks, they may not be willing to buy them at a price $q_0^i = 1$. Selling them at a price less than 1 means that depositors are subsidizing the banks, of course. We therefore assume that it is illegal to sell deposits in this way; all depositors, both consumers in region i or bank from other regions, must be treated in the same way. Therefore, $q_0^i \equiv 1$ and bank deposits can be issued to banks in other regions only if it is optimal for other banks to accept them. In equilibrium, deposits held by banks from other regions are demand-determined.

To begin with, we consider a single bank in region i , and assume that

the bank does not issue any deposits to other banks. The choices of the distinguished bank are marked by a $\hat{\cdot}$ and the usual notation is used for the other banks, in region i and in other regions. Suppose then that the bank has issued one unit of deposits and used the proceeds to invest in a portfolio $(\hat{x}_0^i, \hat{y}_0^i, \hat{z}_0^i)$. It also chooses a deposit contract $(\hat{c}_1^i, \hat{c}_2^i)$. Let $\hat{a}_0^i = (\hat{x}_0^i, \hat{y}_0^i, \hat{z}_0^i, \hat{c}_1^i, \hat{c}_2^i)$ denote the bank's action at date 1. The other banks in region i choose a portfolio (x_0^i, y_0^i, z_0^i) deposit contract (c_1^i, c_2^i) and banks in regions $j \neq i$ choose the portfolio (x_0^j, y_0^j, z_0^j) and deposit contract (c_1^j, c_2^j) . The liquidation values of the other banks are given by q , which is determined in the usual way.

Once ω is realized at date 1, the bank learns whether it is insolvent or not, that is, whether there is a feasible portfolio choice that allows it to pay \hat{c}_1^i to withdrawers at date 1. In any case, there is a well defined liquidation value of deposits $\hat{q}_t^i(\omega)$ at date t and the payoff to depositors is $u(\hat{q}_1^i(\omega))$ if the bank is insolvent at date 1 and

$$\omega^i u(\hat{q}_1^i(\omega)) + (1 - \omega^i) u(\hat{q}_2^i(\omega))$$

if the bank is solvent at date 1. It is important to remember here that \hat{q}_t^i is a function of the initial action \hat{a}_0^i as well as the actions of the representative banks in each region. Let $U^i(\hat{a}_0^i, \omega)$ denote the expected utility in state ω given the action \hat{a}_0^i at date 0. Then the expected utility of the depositors is

$$U(\hat{a}_0^i) = \sum_{\omega \in \Omega} p(\omega) U(\hat{a}_0^i, \omega).$$

The bank is assumed to choose \hat{a}_0^i to maximize $U(\hat{a}_0^i)$.

5 Decentralization of the Social Optimum

The optimal risk-sharing problem (P1) discussed in Section 3 maximizes the unweighted sum of expected utilities. There are other efficient allocations besides the solution to this problem but given the symmetry of the model—regions are ex ante identical—makes this a natural benchmark for the efficiency of risk sharing. In this section, we show that under certain conditions the first best can be decentralized by a competitive banking sector issuing standard deposit contracts. The main assumption we need is that there is no aggregate uncertainty.

- The size of the aggregate shock $\hat{\omega} = \sum_{i=1}^n \omega^i$ is the same for every state $\omega \in \Omega$.

This assumption implies that the consumption allocation corresponding to the solution of (P1) is a constant (c_1, c_2) , independent of the state ω . This is necessary to allow banks to implement the optimal risk sharing through non-contingent deposit contracts.

5.1 Complete Markets

There are no bank runs in an equilibrium that decentralizes the solution of the planner's problem (P1). Each bank has enough assets to provide consumers with the optimal consumption allocation (c_1, c_2) . Deposits in different banks are perfect substitutes and hence have the same value at date 0.

At date 0 an individual bank in region i chooses a portfolio (x_0^i, y_0^i) subject to the feasibility constraint $x_0^i + y_0^i \leq 1$ and offers the depositors in his bank a deposit contract $(c_1^i, c_2^i) = (c_1, c_2)$.

Let z_{jt}^i denote the number of deposits held by the typical bank in region i in banks in region j at the end of date $t = 0, 1$. Suppose that each bank chooses a symmetrical portfolio z_0^i at date 0, where

$$z_{j0}^i = \begin{cases} \zeta & j \neq i \\ 0 & j = i \end{cases},$$

where $\zeta > 0$ is a large number. Each bank in region i chooses the same portfolios

$$(x_t^i, y_t^i) = n^{-1}(x_t, y_t)$$

at dates $t = 0, 1$ where (x_t, y_t) denotes the planner's portfolio at date t in the solution to (P1). Since there is no aggregate uncertainty in the planner's problem, we know that $\hat{\omega}c_1 = y_0$ and $(1 - \hat{\omega})c_2 = Rx_0$, that is, consumption in the intermediate period is financed through the short asset and consumption in the final period is financed through the long asset. This implies that the budget constraint at date 1 requires

$$\begin{aligned} \left[\sum_{j \neq i} (z_{i0}^j - z_{i1}^j(\omega)) - \sum_{j \neq i} (z_{j0}^i - z_{j1}^i(\omega)) \right] c_1 &\leq (y_0^i - y_1^i(\omega)) + r(x_0^i - x_1^i(\omega)) \\ &= y_0^i. \end{aligned} \tag{4}$$

Let $\gamma \equiv n^{-1}\hat{\omega}$ denote the average number of early consumers in each region and note that $y_0^i = \gamma c_1$, so the deficit in region i is $\omega^i - \gamma$ deposits. The budget constraint (4) will be satisfied if and only if

$$\left[\sum_{j \neq i} (z_{i0}^j - z_{i1}^j(\omega)) - \sum_{j \neq i} (z_{j0}^i - z_{j1}^i(\omega)) \right] = \omega^i - \gamma.$$

In region i put $z_1^i(\omega) = z_0^i$ if $\omega^i - \gamma \leq 0$. For any region i such that $\omega^i - \gamma > 0$ let $\eta^i(\omega)$ denote the ratio of region i 's excess demand to total excess demand:

$$\eta^i(\omega) = \frac{\omega^i - \gamma}{\sum_i \max\{\omega^i - \gamma, 0\}}.$$

Then define region i 's portfolio $z_1^i(\omega)$ by putting

$$z_{j1}^i(\omega) = \begin{cases} 0 & \text{if } \omega^j - \gamma > 0 \\ z_0^i - \eta^i(\omega)(\omega^j - \gamma) & \text{if } \omega^j - \gamma \leq 0. \end{cases}$$

In words, if the banks in region i need extra liquidity, they do not draw down deposits in banks in regions j that are liquidity-constrained ($\omega^j - \gamma > 0$). Instead, they proportionately draw down their deposits in regions that have excess liquidity ($\omega^j - \gamma \leq 0$). It is clear that the portfolios defined satisfy the budget constraint (4). Obviously, there are many other ways of defining a portfolio $z_1^i(\omega)$ to satisfy the budget constraint.

Summing the budget constraints (4) and making use of the planners budget constraint for date 1 we end up with the interbank deposit market-clearing condition at date 1

$$\sum_i \left[\sum_{j \neq i} (z_{i0}^j - z_{i1}^j(\omega)) - \sum_{j \neq i} (z_{j0}^i - z_{j1}^i(\omega)) \right] = 0.$$

At date 2 the budget constraint for a bank in region i will be

$$\left[\sum_{j \neq i} z_{i1}^j(\omega) - \sum_{j \neq i} z_{j1}^i(\omega) + (1 - \omega^i) \right] c_2 \leq y_1^i(\omega) + Rx_1^i(\omega). \quad (5)$$

Summing (5) across i we obtain the planner's budget constraint for date 2 if and only if

$$\sum_i \left[\sum_{j \neq i} z_{i1}^j(\omega) - \sum_{j \neq i} z_{j1}^i(\omega) \right] = 0.$$

The market-clearing condition in deposits for date 2 is identically zero. It is easy to check that if banks satisfy the budget constraint (4) then the budget constraint (5) is automatically satisfied. Given a sequence of portfolios (z_0^i, z_1^i) satisfying the budget constraints (4) and (5) for each region i , the banks can achieve the same investment portfolios and consumption allocations for their depositors as the central planner. It remains to check that this is an equilibrium.

At date 2 there are no decisions to be made. The value of deposits in region i will be $q_2^i(\omega) = c_2^i = c_2$ in every state of nature ω .

At date 1 the value of deposits will be $q_1^i(\omega) = c_1^i = c_1$ in every state of nature ω and since $c_1 \leq c_2$ it is optimal for early and late consumers to withdraw at date 1 and date 2 respectively.

Once the state ω is revealed at date 1, it is clear that each bank cannot do better than to meet its obligations under the deposit contract (c_1, c_2) . There is no portfolio decision that allows the bank to meet its obligations and achieve a surplus in either period. At date 0 things are more complicated. We assume that each bank wants to maximize the expected utility of the typical consumer subject to the budget constraints at dates $t = 1, 2$. The question then is whether it can do better than the choice of portfolios and deposit contract described above.

For the case of an equilibrium with complete risk sharing, it is easy to characterize the optimality of bank's behavior. Suppose that all banks in region i choose the deposit contract (c_1^i, c_2^i) and the sequence of portfolios $\{(x_t^i, y_t^i, z_t^i)\}_{t=0,1}$ and a single bank in some region k deviates by choosing a contract (c_1', c_2') and a sequence of portfolios $\{(x_t', y_t', z_t')\}_{t=0,1}$. We can treat the resulting re-allocation as the result of a trade between the deviating bank and a fully-insured representative bank in each region i . The effect of the deviating bank choosing a different portfolio and deposit contract is to effect a contingent transfer of consumption between the deviating bank and the representative banks in each region. The equilibrium deposit contract (c_1, c_2) solves the maximization problem

$$\begin{aligned} \hat{\omega}u(c_1) + (1 - \hat{\omega})u(c_2) &= \sup_{\hat{\omega}c_1 + (1 - \hat{\omega})R^{-1}c_2 \leq 1} \{\hat{\omega}u(c_1) + (1 - \hat{\omega})u(c_2)\} \\ &= \sup_{\omega^i c_1 + (1 - \omega^i)R^{-1}c_2 \leq 1} \sum_{\omega} p(\omega) \{\omega^i u(c_1(\omega)) + (1 - \omega^i)u(c_2(\omega))\}. \end{aligned}$$

The representative bank in region i will not accept deposits in the deviating bank unless they leave depositor welfare at least as high as before. This

means that

$$\sum_{\omega} p(\omega) \left\{ \omega^i u'(c_1) \frac{\Delta_1^i(\omega)}{\omega^i} + (1-\omega^i) u'(c_2) \frac{\Delta_2^i(\omega)}{1-\omega^i} \right\} = u'(c_1) \sum_{\omega} p(\omega) \{ \Delta_1^i(\omega) + R^{-1} \Delta_2^i(\omega) \} \geq 0,$$

where $\Delta_t^i(\omega)$ is the net transfer to region i at date t as a result of the deviating bank's trade. But this means that

$$\sum_{\omega} p(\omega) \left\{ \omega^k u'(c_1) \frac{\Delta_1(\omega)}{\omega^k} + (1-\omega^k) u'(c_2) \frac{\Delta_2(\omega)}{1-\omega^k} \right\} = u'(c_1) \sum_{\omega} p(\omega) \{ \Delta_1(\omega) + R^{-1} \Delta_2(\omega) \} \leq 0,$$

where

$$\Delta_t(\omega) = - \sum_{i=1}^n \Delta_t^i(\omega),$$

for $t = 1, 2$. In other words, the depositors of the deviating bank cannot be made better off by any feasible state-contingent transfers that are acceptable to the other banks. The fact that the deviating bank cannot make arbitrary state-contingent re-allocations, but has to use deposit contracts, simply restricts its ability to increase welfare. The Pareto-optimality of (c_1, c_2) makes it clear that it is impossible for a deviating bank to make itself better off through trade with the representative bank.

Theorem 2 *Let $(x_0, y_0, x_1(\omega), y_1(\omega), c_1, c_2)$ be the solution to the planner's problem (P1) and suppose that the representative bank in each region i chooses a deposit contract $(c_1^i, c_2^i) = (c_1, c_2)$, a sequence of portfolios $(x_0^i, y_0^i, z_0^i, x_1^i(\omega), y_1^i(\omega), z_1^i(\omega))$ satisfying*

$$(x_0^i, y_0^i, x_1^i(\omega), y_1^i(\omega)) = n^{-1}(x_0, y_0, x_1(\omega), y_1(\omega)),$$

and the budget constraints (4) and (5). Then the equilibrium of the decentralized banking system described above implements the first-best allocation.

5.2 Incomplete Markets

In constructing an equilibrium in which the social optimum could be decentralized, it was assumed that each bank can hold a complete portfolio of deposits. In practice this is unlikely to be true, because of the complexity of such a strategy or because of the informational requirements. However, complete markets are not necessary in order to decentralize the first best as long as the regions are connected in an appropriate sense. As an illustration,

consider the case where regions are arranged clockwise in a circular network. Each region i is directly connected to the next region $i + 1$ in the clockwise direction, that is, for each region i the neighboring set is $N^i = \{i + 1\}$ where region $n + 1$ is identified with region 1. With this restriction, the overall structure of equilibrium is similar to that described in the previous section.

To show that social optimum can be decentralized with this market structure, it is sufficient to show that the banks can satisfy their budget constraints. At date 0 each bank chooses an admissible portfolio

$$z_{j0}^i = \begin{cases} \zeta & \text{if } j = 1 + 1; \\ 0 & \text{otherwise.} \end{cases}$$

At date 1 the bank has to choose an admissible portfolio $z_1^i(\omega)$ such that

$$\left[\sum_{j=i\pm 1} (z_{i0}^j - z_{i1}^j(\omega)) - \sum_{j=i\pm 1} (z_{j0}^i - z_{j1}^i(\omega)) + \omega^i \right] c_1 \leq y_0^i.$$

If this constraint is satisfied then the constraint at date 2 is automatically satisfied. The aggregate budget constraint at date 1 $\hat{\omega} c_1 = y_0$ ensures that we can find portfolios $\{z_1^i\}$ satisfying the regional budget constraints if ζ is chosen large enough. This is all one needs to decentralize the social optimum.

For the general case of an arbitrary, connected collection of neighborhoods $\{N^i\}$, we can show that it is possible to decentralize the the social optimum by constructing a set of deposit portfolios $\{z_t^i\}$ that will satisfy the budget constraints above. The easiest way to do this is to begin with the equilibrium portfolios for the economy with complete markets and then construct an equivalent profile of portfolios for the economy with neighborhoods $\{N^i\}$. Let $\{z_t^i\}$ denote the portfolios for the complete-markets economy and let $\{\hat{z}_t^i\}$ denote the corresponding portfolios for the incomplete-markets economy. To construct $\{\hat{z}_t^i\}$ we proceed as follows. Take $i = 1$ and some $t = 0$ and consider the smallest index $j \neq i$ such that $z_{j0}^i \neq 0$ and region i cannot hold deposits in region j . Then the connectedness assumption implies that there exists a chain $\{i_1, \dots, i_K\}$ such that i_k is directly connected to i_{k+1} and $i_1 = i$ and $i_K = j$. Then let $\hat{z}_{j0}^i = 0$ and put $\hat{z}_{i_{k+1}0}^i = z_{i_{k+1}0}^i + z_{j0}^i$ for every $k = 1, \dots, K - 1$. Replace z_{j0}^i with \hat{z}_{j0}^i and apply the same procedure again until the portfolio for region i is admissible. Then move on to the next index $i = 2$ and so on until all the regions have admissible portfolios at date 0. Then go through the same procedure at date $t = 1$ for each state ω and region i .

This procedure clearly results in a set of admissible profiles $\{z_t^i\}$ for all regions and dates $t = 0, 1$. Also, it is easy to see that these portfolios are equivalent to the original portfolios, in the sense that each region has the same number of deposits as before in every state and at every date. For this reason, the new portfolios must satisfy the budget constraints at each date. Another way to see that the budget constraints must be satisfied is to note that the changes we have made simply create a series of “pass throughs”, where region $i = i_1$ holds a deposit in region j indirectly by holding a deposit in region i_2 which holds an offsetting deposit in region $i_3 \dots$ which holds an offsetting deposit in region $i_K = j$.

Thus, the decentralization argument given in the first part of this section extends easily to any connected family of neighborhoods $\{N^i\}$.

6 Contagion

To illustrate the possibility of contagion, we use the decentralization theorems from Section 5 to show the existence of an equilibrium with complete risk sharing. Then we perturb the model to show that for some states a small excess demand for liquidity can lead to an economy-wide crisis. In other words, the equilibrium with complete risk sharing suffers from financial fragility.

In Section 5 we showed that complete risk sharing could be decentralized using standard deposit contracts if (a) there is no aggregate risk, that is, if $\hat{\omega}$ was constant and (b) the economy is connected, that is, the family of neighborhoods $\{N^i\}$ is connected. As an illustration of the contagion problem, we take a particular structure of the admissible deposit portfolio sets, namely, the case in which each region i can hold deposits only in region $i + 1$, where we identify region $n + 1$ with region 1. This would be the case if the regions were arranged clockwise in a circle, with banks in region i being able to hold deposits in the neighboring region in the clockwise direction.

Let (c_1, c_2) denote the optimal consumption profile chosen by a central planner when there is no aggregate uncertainty. Let (x, y) denote the per capita investments in the long and short assets, respectively, chosen by the planner. The symmetry of the model implies that there will be an equilibrium, in which every region behaves symmetrically in the first period. The optimal deposit contract in region i will be

$$(c_1^i, c_2^i) = (c_1, c_2).$$

The initial portfolio will be

$$(x_0^i, y_0^i, z_0^i) = (x, y, z_0^i)$$

where

$$z_{j0}^i = \begin{cases} \zeta & j = i + 1 \\ 0 & j \neq i + 1 \end{cases}$$

and ζ is an appropriately chosen constant. We choose ζ to be as small as possible, but assuming that there is non-degenerate uncertainty ζ must be positive in equilibrium.

At date 1 the state ω is observed and all the early consumers (and only the early consumers) withdraw their deposits worth c_1 each. In order to satisfy the budget constraint, banks in region i choose (x_1^i, y_1^i, z_1^i) so that

$$(x_1^i(\omega), y_1^i(\omega)) = (x_0^i, 0)$$

(there is no need to liquidate the long asset and it is never optimal to carry the short asset over to the last period) and

$$\omega^i c_1 = y_0^i + (z_{i+1,0}^i - z_{i+1,1}^i(\omega))c_1 - (z_{i0}^{i-1} - z_{i1}^{i-1}(\omega))c_1.$$

In the last period, the budget constraint is

$$(1 - \omega^i)c_2 = Rx_0^i + z_{i+1,1}^i(\omega)c_2 - z_{i1}^{i-1}(\omega)c_2.$$

The budget constraint at date 2 is satisfied if the budget constraint at date 1 is satisfied, so it is enough to show that there exists a choice of z_1^i for each i that satisfies the budget constraint at date 1. The existence of these portfolios $\{z_1^i\}$ follows from arguments provided in Section 5. Here we note that there is an essentially unique way of defining the portfolios. Suppose that $z_{2,1}^1(\omega)$ is given for some state ω . Then the second-period budget constraint for $i = 2$ determines $z_{3,1}^2(\omega)$. Continuing in this way we can show that if $z_{i+1,1}^i$ is given then the second-period budget constraint for $i + 1$ determines $z_{i+2,1}^{i+1}$. When we get back to region $i = 1$, the value of $z_{2,1}^1(\omega)$ determined by the budget constraint must agree with the value initially given, because summing the budget constraints from $i = 2, \dots, n$ gives

$$\sum_{i=2}^n \omega^i c^1 = (n-1)y_0 + (z_{1,0}^n - z_{1,1}^n(\omega))c_1 - (z_{2,0}^1 - z_{2,1}^1(\omega))c_1$$

which implies, on substituting $\hat{\omega} = \sum_{i=1}^n \omega^i$,

$$(\hat{\omega} - \omega^1)c^1 = (n-1)y_0 + (z_{1,0}^n - z_{1,1}^n(\omega))c_1 - (z_{2,0}^1 - z_{2,1}^1(\omega))c_1$$

or equivalently

$$\omega^1 c^1 = y_0 + (z_{2,0}^1 - z_{2,1}^1(\omega))c_1 - (z_{1,0}^n - z_{1,1}^n(\omega))c_1.$$

This shows that the originally given value of $z_2^1(\omega)$ satisfies the budget constraint for region $i = 1$.

We have not shown that the portfolios defined in this way satisfy the non-negativity constraint. However, by choosing $z_{2,1}^1$ sufficiently large, all of the other values will be non-negative. So choose $z_{2,1}^1$ to be the smallest value that is consistent with the non-negativity constraint.

It is straightforward to check that the portfolio defined in this way is the (essentially) unique portfolio that satisfies the budget constraint in both periods.

Note that there is an indeterminacy in the definition of the portfolio in the initial period because we have not specified ζ . Clearly, ζ has to be big enough to allow for the definition of a non-negative z_1^i at date 1. Let ζ be chosen to be the smallest value of ζ that is consistent with a non-negative z_1^i . With this convention, the equilibrium is uniquely defined. Theorem 2 shows that this is in fact an equilibrium.

Now, let us take the equilibrium as given and consider what happens when we “perturb” the model. By a perturbation I mean the realization of a state $\bar{\omega}$ that was assigned zero probability at date 0 and has a demand for liquidity that is very close to that of the states that do occur with positive probability. Specifically, define the state $\bar{\omega}$ by putting

$$\bar{\omega}^i = \begin{cases} \gamma & i \neq k \\ \gamma + \varepsilon & i = k \end{cases}.$$

Thus, at date 0 the choices of deposit contract (c_1^i, c_2^i) and initial portfolio (x_0^i, y_0^i, z_0^i) are the same as in the equilibrium with complete risk sharing. Furthermore, for any of the states ω that occur with positive probability, the equilibrium proceeds at dates 1 and 2 in the way described above. For the distinguished state $\bar{\omega}$ things are different, as we show in a number of steps.

Step 1. In state $\bar{\omega}$ there must be at least one region in which the banks are insolvent. The proof is by contradiction. Suppose, contrary to what we want

to prove, that banks in all regions are solvent. Then $q^i(\bar{\omega}) = c_1$ for all i . However, the demand for deposits from early consumers is γ in each region and the stock of the short asset is $y_0^i = \gamma c_1$, so there is an excess demand for liquidity that can only be met by liquidating the long asset in some region. Any bank that liquidates the long asset will lose value. More precisely, the bank has just enough of the short asset to meet the demands of its local depositors (early consumers), so it has to liquidate some of the long assets if it allows other banks to withdraw more from it than it withdraws from them. For every unit of the long asset it liquidates it gives up R future units of the good and gets r present units of the good. For every unit of deposits it retains it gets c_2 future units of the good and gives up c_1 present units of the good. So it is costly to liquidate the long asset if

$$\frac{R}{r} > \frac{c_2}{c_1}. \quad (6)$$

This inequality will hold, for example, if $u'(c)c$ is decreasing because that implies that

$$\frac{u'(c_1)c_1}{u'(c_2)c_2} > 1 \quad (7)$$

and the optimal deposit satisfies

$$u'(c_1) = Ru'(c_2) \quad (8)$$

so using the inequalities (7) and (8) we have

$$\frac{c_2}{c_1} < \frac{u'(c_1)}{u'(c_2)} = R < \frac{R}{r}$$

as required. The assumption (6) is maintained in what follows, so no bank will willingly liquidate the long asset.

To avoid liquidating the long asset, banks must redeem at least as many deposits as are withdrawn by banks from other regions. But this implies that no region is able to get extra liquidity from other regions. The only equilibrium is one in which all banks simultaneously withdraw their deposits in banks in other regions at date 1 and these mutual withdrawals offset each other and so have no effect. The result is that banks in region k are forced to be self-sufficient. Then solvency in region k requires

$$\begin{aligned} (\gamma + \varepsilon)c_1 &\geq y_0^k + r(x_0^k - x_1^k(\bar{\omega})) \\ (1 - \gamma - \varepsilon)c_1 &\geq Rx_1^k(\bar{\omega}). \end{aligned} \quad (9)$$

The demand for deposits is $\gamma + \varepsilon$ and solvency requires that each agent should get c_1 . The short asset y_0^k will be used first and then some of the long asset will be liquidated to yield an additional $r(x_0^k - x_1^k(\bar{\omega}))$. At the last date, the late consumers must be given at least c_1 to prevent a run, so the amount paid out will be at least $(1 - \gamma - \varepsilon)c_1$, and the liquidation value of the bank's portfolio will be $Rx_1^k(\bar{\omega})$. The conditions in equation (9) are necessary and sufficient for solvency at date 1, so if these conditions are violated then banks in region k must be insolvent and a bank run (crisis) occurs. In what follows we assume that equation (9) is violated.

Note that we have not yet shown that region k must be insolvent; only that some region must be insolvent. However, it is easy to see that in any region other than region k the banks can protect themselves against insolvency by liquidating all their deposits in other regions' banks at date 1, on the assumption that the late consumers do not run unless there is no equilibrium in their region in which it is optimal for them to withdraw late. Hence the only equilibrium will be one in which banks in region k (and possibly other regions) are insolvent.

Step 2. Having established that banks in region k must be insolvent, we next show that the financial crisis must be extend to other regions. We assume to the contrary that all regions $i \neq k$ are solvent. Since the banks in region k are insolvent, they must liquidate all their holdings of deposits in region $k + 1$ and the banks in region $k - 1$ will find it optimal to liquidate their deposits in banks of region k . The value of the deposits in region k are determined by the condition that the value of liabilities equals the value of assets:

$$(1 + \zeta)q_1^k(\bar{\omega}) = y_0^k + rx_0^k + \zeta c_1$$

since the value of deposits in region $k + 1$ equals c_1 as long as they are solvent. Then

$$q_1^k(\bar{\omega}) = \frac{y_0^k + rx_0^k + \zeta c_1}{1 + \zeta}$$

and the transfer from banks in region $k + 1$ to banks in region k is

$$\begin{aligned} \zeta(c_1 - q_1(\bar{\omega})) &= \frac{\zeta(1 + \zeta)}{1 + \zeta}c_1 - \frac{\zeta}{1 + \zeta}(y_0^k + rx_0^k + \zeta c_1) \\ &= \frac{\zeta(c_1 - y_0^k - rx_0^k)}{1 + \zeta}. \end{aligned}$$

By a previous argument, the banks in regions $i \neq k, k - 1$ will not want to liquidate the long asset, so without loss of generality, we can assume that

they all liquidate their holdings of deposits in other regions in the current period. This means that banks in region $k - 1$ have to remain self-sufficient. Solvency in this region is only possible if there is a solution to

$$\begin{aligned}\gamma c_1 &\leq y_0^j + r(x_0^j - x_1^j(\bar{\omega})) - \frac{\zeta(c_1 - y_0^k - rx_0^k)}{1 + \zeta} \\ (1 - \gamma)c_1 &\leq Rx_1^j(\bar{\omega}).\end{aligned}$$

for $j = k - 1$. Since $\gamma c_1 = y_0^j$ and $(1 - \gamma)c_2 = Rx_0^j$, these conditions can be satisfied if and only if

$$\frac{\zeta(c_1 - y_0^k - rx_0^k)}{1 + \zeta} \leq r(x_0^j - x_1^j(\bar{\omega})) \leq \frac{r(1 - \gamma)(c_2 - c_1)}{R}.$$

The last expression on the right is the amount of liquidity that we can get at date 1 without violating the incentive constraint at date 2. Making similar substitutions, the first expression on the left can be rewritten as

$$\frac{\zeta((1 - \gamma)c_1 - r(1 - \gamma)c_2/R)}{1 + \zeta}$$

so the inequality becomes

$$\frac{\zeta(c_1 - rc_2/R)}{1 + \zeta} \leq \frac{r(c_2 - c_1)}{R}$$

or

$$\frac{\zeta(Rc_1 - rc_2)}{1 + \zeta} \leq r(c_2 - c_1)$$

which is certainly violated for r small enough, for example.

Step 3. The preceding step has shown that under certain conditions we must have bank runs in equilibrium in region $k - 1$. To ensure that the process continues to infect the other regions we only need to show that the value of the bank deposits in region $k - 1$ will be even lower than we assumed in region k . To see this, note that once a run has occurred, the banks in region $k - 1$ have even less assets than the banks in region k in the previous step. Whereas the banks in region k could call upon the solvent banks in region $k + 1$, whose deposits are worth c_1 , the banks in region $k - 1$ can only call on the insolvent banks in region k , whose deposits are worth $q_1(\bar{\omega}) < c_1$. Their demand for liquidity is the same, so they are in a worse position than the

banks in region k . As a result, $q_{k-1}(\bar{\omega}) < q_k(\bar{\omega}) < c_1$, so region $k + 2$ must be insolvent. As we continue to argue by induction, at each step the value of deposits in the marginal insolvent regions gets lower and it is easier to prove that the marginal solvent regions cannot satisfy the conditions for solvency. We conclude that all regions must be insolvent. Note that this implies that the values of deposits in all regions are the same: $q^i(\bar{\omega}) = y_0^i + rx_0^i$.

7 Robustness of Complete Markets

The incompleteness of markets is essential to the contagion result in the following sense. There exist parameter values for which any equilibrium with incomplete markets involves runs in state $\bar{\omega}$ (this is the set of parameter values characterized in Section 6). For the same parameter values, we can find an equilibrium with complete markets that does not involve runs in state $\bar{\omega}$.

To see this, we go back to the complete markets equilibrium in Section 5 under the assumption that $\hat{\omega}$ is a constant. In that case, the non-contingent deposit contract (c_1, c_2) is the first best and we have seen that there is a sequence of portfolios $\{(x_0^i, y_0^i, z_0^i), (x_1^i, y_1^i, z_1^i)\}$ that implements this contract as an equilibrium. Now suppose that we introduce the small probability state $\bar{\omega}$ that led to contagion in Section 6. Of course, several restrictions on the parameter values were required in order to generate runs and those restrictions are assumed to be satisfied here. The question is whether there exists an equilibrium for state $\bar{\omega}$ in which runs do not occur, even though those conditions are satisfied? To answer this question in the affirmative, we have to show that it is possible to liquidate deposits without violating the conditions

$$q_1^i(\bar{\omega}) = c_1^i \leq q_2^i(\bar{\omega}), \quad (10)$$

for $i \neq k$. As long as these conditions are satisfied, there will not be any runs in the regions $i \neq k$.

Without loss of generality, we consider the case in which $q_1^k(\bar{\omega}) < c_1^k$. Otherwise, there is no difficulty in showing that every region is solvent. Then banks in region k will liquidate all of their claims against banks in regions $i \neq k$ and banks in regions $i \neq k$ will liquidate all their deposits in region k . In order to satisfy the conditions 10, banks in regions $i \neq k$ must be able to

find a portfolio $(x_1^i(\bar{\omega}), y_1^i(\bar{\omega}), z_1^i(\bar{\omega}))$ such that

$$(\gamma + \zeta)c_1^i \leq y_0^i + r(x_0^i - x_1^i(\bar{\omega})) + q_1^k(\bar{\omega})\zeta$$

and

$$(1 - \gamma)c_1^i \leq Rx_1^i(\bar{\omega}).$$

We have already seen that these inequalities cannot be satisfied for the value ζ required in the equilibrium with incomplete markets. But the value of ζ required here is different. With incomplete markets, the typical bank holds deposits in two regions; with complete markets it holds deposits in $n - 1$ regions. Therefore the value of ζ in equilibrium with complete markets is $2/(n - 1)$ times the value required in equilibrium with incomplete markets. Since there is no constraint on how large n can be, we can certainly find parameter values for which the solvency conditions (10) are satisfied for complete markets but not for incomplete markets.

Another way of stating this is that for any definition of $\bar{\omega}$ (any feasible choice of $\varepsilon > 0$), the complete markets equilibrium is robust for n large enough, whereas for some parameter values the incomplete markets equilibrium will not be.

8 Containment

The critical ingredient in the example of contagion analysed in Section 6 is that any two regions are connected by a chain of overlapping bank liabilities. Banks in region i have claims on banks in regions $i - 1$ and $i + 1$, which in turn have claims on banks in regions $i - 2$ and $i + 2$, respectively, and so on. If we could cut this chain at some point, the contagion that begins with a small shock in region k would be contained in some connected component of the set of regions. The structure of claims is endogenous, however, so we cannot simply assume that the whole economy is not enmeshed in a single web of claims. Some restrictions on the structure of the model are required in order to ensure that the economy can be broken down into small independent clumps of regions. One way to do this is to assume a special structure of liquidity shocks, which does not require much interconnectedness of claims in order to achieve complete risk sharing.

As an illustration, consider the special case in which adjacent pairs of regions can achieve complete risk sharing. More precisely, assume that for

some constant γ

$$\omega^i + \omega^{i+1} = 2\gamma \quad (11)$$

for every region i and almost every state ω . Equation (11) implies that the number of regions n is an even number, of course. An equilibrium is defined by analogy with the equilibrium in Section 6 except that claims are exchanged between pairs of regions only. Let (x_0, y_0) be the initial portfolio and (c_1, c_2) be the deposit contract chosen by the central planner as the solution to (P1). This allocation can be decentralized with incomplete markets as follows. At date 0, the representative bank in each region i chooses a portfolio (x_0^i, y_0^i, z_0^i) and a deposit contract (c_1^i, c_2^i) , where

$$(x_0^i, y_0^i) = n^{-1}(x_0, y_0)$$

and

$$(c_1^i, c_2^i) = (c_1, c_2).$$

The main difference from the equilibrium described in Section 6 is that the deposits are exchanged between pairs of regions: if i is an even number, then regions i and $i + 1$ exchange claims, but regions i and $i - 1$ do not. Thus,

$$z_{j0}^i = \begin{cases} \zeta & i \text{ is even and } j = i + 1; \\ \zeta & \text{if } i \text{ is odd and } j = i - 1; \\ 0 & \text{otherwise.} \end{cases}$$

At date 1 the state ω is observed and each pair of regions $(i, i + 1)$, where i is an even number, adjust their holdings of deposits so that the optimal consumption allocation can be achieved. Specifically, $q_1^i(\omega) = c_1$ for every region i and any state ω and

$$(x_1^i(\omega), y_1^i(\omega), z_1^i(\omega)) = (x_0^i, 0, z_1^i(\omega)),$$

where $z_1^i(\omega)$ is chosen so that

$$z_{j1}^i(\omega) = \begin{cases} z_{j0}^i - (\omega^i - \gamma) & \text{if } i \text{ is even and } j = i + 1; \\ z_{j0}^i - (\omega^i - \gamma) & \text{if } i \text{ is odd and } j = i - 1; \\ 0 & \text{otherwise.} \end{cases}$$

At date 2 the banks liquidate their remaining portfolios and it is easy to see from the banks' budget constraint that the value of deposits will be $q_2^i(\omega) = c_2^i$ in every region.

It is straightforward to show, in the same way as in Section 6, that these choices are optimal and feasible, in other words, that they constitute an equilibrium.

Now suppose that we perturb the equilibrium by introducing a small probability state $\bar{\omega}$ in which there is an excess demand for liquidity in some state k :

$$\bar{\omega}^i = \begin{cases} \gamma + \varepsilon & \text{if } i = k; \\ \gamma & \text{if } i \neq k. \end{cases}$$

Without loss of generality we can assume that k is even; the other case is exactly symmetric. Then regions k and $k+1$ are linked by overlapping claims to deposits, but there are no claims linking regions $k-1$ and k or linking regions $k+1$ and $k+2$. Thus, the component $(k, k+1)$ is independent of the remainder of the regions. In fact, it is clear that *whatever happens in regions k and $k+1$* , we can define an equilibrium for regions $i \neq k, k+1$ in which $q_t^i(\bar{\omega}) = c_t^i$ for $t = 1, 2$. In words, in state $\bar{\omega}$ there may be bank runs in region k and they may spill over to region $k+1$, but this will have no effect on other regions. The effects of the disturbance associated with state $\bar{\omega}$ will be contained in the component $(k, k+1)$.

The same general result will hold whenever we can establish that a set of regions $C \subset \{1, \dots, n\}$ is disconnected from its complement in equilibrium.

9 Conclusion

In this paper, we have considered a more general version of the model in Allen and Gale (2000), which only considered networks of four banks. We show that similar results hold in much more general contexts with an unlimited number of banks. In particular shocks that are small relative to the economy as a whole can cause a collapse of the financial system. There are two main differences with the Diamond-Dybvig model. The first is the assumption that the illiquid, long-term assets held by the banks are risky and perfectly correlated across banks. Uncertainty about asset returns is intended to capture the impact of the business cycle on the value of bank assets. Information about returns becomes available before the returns are realized and when the information is bad it has the power to precipitate a crisis. The second is that we do not make the first-come, first-served assumption. This assumption has been the subject of some debate in the literature as it is not an optimal arrangement in the basic Diamond-Dybvig model (see

Wallace (1988) and Calomiris and Kahn (1991)). In a number of countries and historical time periods banks have had the right to delay payment for some time period on certain types of account. This is rather different from the first-come, first-served assumption. Sprague (1910) recounts how in the U.S. in the late nineteenth century people could obtain liquidity once a panic had started by using certified checks. These checks traded at a discount. We model this type of situation by assuming the available liquidity is split on an equal basis among those withdrawing early. In the context this arrangement is optimal. We also assume that those who do not withdraw early have to wait some time before they can obtain their funds and again what is available is split between them on an equal basis.

A number of authors have developed models of banking panics caused by aggregate risk. Wallace (1988; 1990), Chari (1989) and Champ, Smith, and Williamson (1996) extend Diamond and Dybvig (1983) by assuming the fraction of the population requiring liquidity is random. Chari and Jagannathan (1988), Jacklin and Bhattacharya (1988), Hellwig (1994), and Alonso (1996) introduce aggregate uncertainty which can be interpreted as business cycle risk. Chari and Jagannathan (1988) focus on a signal extraction problem where part of the population observes a signal about future returns. Others must then try to deduce from observed withdrawals whether an unfavorable signal was received by this group or whether liquidity needs happen to be high. Chari and Jagannathan are able to show panics occur not only when the outlook is poor but also when liquidity needs turn out to be high. Jacklin and Bhattacharya (1988) also consider a model where some depositors receive an interim signal about risk. They show that the optimality of bank deposits compared to equities depends on the characteristics of the risky investment. Hellwig (1994) considers a model where the reinvestment rate is random and shows that the risk should be born both by early and late withdrawers. Alonso (1996) demonstrates using numerical examples that contracts where runs occur may be better than contracts which ensure runs do not occur because they improve risk sharing.

Another important feature of the model is the fact that connections between regions take the form of interbank deposits rather than investments in real assets. Of course, since the motive for investing in other regions is to provide insurance against liquidity shocks, it is natural to assume that banks have claims on each other. But there might also be a motive for spreading investments across regions if the long asset were risky and the returns were imperfectly correlated across regions. In this case, investing directly in real

assets in other regions would be an alternative to investing in deposits in the banks of other regions. The effects would be quite different, however. When a bank becomes insolvent, it is forced to liquidate its assets, with a consequent loss of value. Another bank which has a claim on the first suffers from this loss of value too, even if the second bank is solvent. If the second bank had invested in a real asset instead, it would have had another option. It could have held the real asset until maturity and thus avoided the loss of value that results from liquidation.

10 Proofs

Theorem 1: *The planner's risk-sharing problem (P1) is equivalent to the modified problem (P2). From this it follows that the solution to the planner's problem is first-best efficient, that is, the incentive constraints do not bind.*

Proof. Consider first the consumption of early consumers. For any state ω with $\hat{\omega} > 0$, concavity of the utility function $u(\cdot)$ implies that

$$\frac{\sum_{i=1}^n \omega^i u(c_1^i(\omega))}{\sum_{i=1}^n \omega^i} \leq u(\hat{c}_1(\omega))$$

where

$$\hat{c}_1(\omega) \equiv \frac{\sum_{i=1}^n \omega^i c_1^i(\omega)}{\sum_{i=1}^n \omega^i}.$$

So there is no loss of generality in assuming that

$$c_1^i(\omega) = \hat{c}_1(\omega),$$

for every i and every ω . A similar argument shows that there is no loss of generality in assuming that

$$c_2^i(\omega) = \hat{c}_2(\omega),$$

for every i and every ω . Finally, suppose that there exist two states ω and ω' such that $\hat{\omega} = \hat{\omega}'$. Then define

$$(c_1(\omega), c_2(\omega)) = (c_1(\omega'), c_2(\omega')) = \frac{1}{2} [(\hat{c}_1(\omega), \hat{c}_2(\omega)) + (\hat{c}_1(\omega'), \hat{c}_2(\omega'))].$$

It is easy to see that $(c_1(\omega), c_2(\omega))$ and $(c_1(\omega'), c_2(\omega'))$ satisfy the feasibility constraints and, because the objective function satisfies

$$\hat{\omega} u(\hat{c}_1(\omega)) + (1 - \hat{\omega}) u(\hat{c}_2(\omega)) = \hat{\omega}' u(\hat{c}_1(\omega')) + (1 - \hat{\omega}') u(\hat{c}_2(\omega')),$$

it follows from the concavity of $u(\cdot)$ that

$$\begin{aligned}\hat{\omega}u(c_1(\omega)) + (1 - \hat{\omega})u(c_2(\omega)) &= \hat{\omega}'u(c_1(\omega')) + (1 - \hat{\omega}')u(c_2(\omega')) \\ &\geq \hat{\omega}u(\hat{c}_1(\omega)) + (1 - \hat{\omega})u(\hat{c}_2(\omega)) \\ &= \hat{\omega}'u(\hat{c}_1(\omega')) + (1 - \hat{\omega}')u(\hat{c}_2(\omega')).\end{aligned}$$

With a slight abuse of notation we write $p(\hat{\omega})$ for the probability of $\hat{\omega}$. Then the planner has to choose a total investment x_0 in the long asset, a total investment y_0 in the short asset, an amount of the long asset $x_1(\hat{\omega})$ to carry through to date 2, an amount $y_1(\hat{\omega})$ of the short asset to carry through to date 2, the consumption $c_1(\hat{\omega})$ of an early consumer, and the consumption $c_2(\hat{\omega})$ of a late consumer in order to maximize the typical consumer's expected utility. Note that the initial investment portfolio (x_0, y_0) does not depend on $\hat{\omega}$ because the planner does not yet know the value of $\hat{\omega}$. However, all the decisions made at date 1 and date 2 depend on $\hat{\omega}$, which is revealed at the beginning of date 1.

The modified risk-sharing problem can be written in per capita terms as follows:

$$(P2) \begin{cases} \max & \sum_{\hat{\omega}} p(\hat{\omega}) \{ \hat{\omega}u(c_1(\hat{\omega})) + (1 - \hat{\omega})u(c_2(\hat{\omega})) \} \\ \text{s.t.} & \text{(i)} \quad x_0 + y_0 \leq n; \\ & \text{(ii)} \quad \hat{\omega}c_1(\hat{\omega}) \leq y_0 - y_1(\hat{\omega}) + r(x_0 - x_1(\hat{\omega})); \\ & \text{(iii)} \quad (1 - \hat{\omega})c_2(\hat{\omega}) \leq y_1(\hat{\omega}) + Rx_1(\hat{\omega}). \end{cases}$$

Suppose that we solve the problem for a particular state $\hat{\omega}$, that is, we solve the problem

$$\begin{aligned} \max & \quad \hat{\omega}u(c_1(\hat{\omega})) + (1 - \hat{\omega})u(c_2(\hat{\omega})) \\ \text{s.t.} & \quad \text{(ii)} \quad \hat{\omega}c_1(\hat{\omega}) \leq y_0 - y_1(\hat{\omega}) + r(x_0 - x_1(\hat{\omega})); \\ & \quad \text{(iii)} \quad (1 - \hat{\omega})c_2(\hat{\omega}) \leq y_1(\hat{\omega}) + Rx_1(\hat{\omega}). \end{aligned}$$

A necessary condition for an optimum is that

$$u'(c_1(\hat{\omega})) \geq u'(c_2(\hat{\omega})),$$

with strict equality if $y_1(\hat{\omega}) > 0$, which implies that

$$c_1(\hat{\omega}) \leq c_2(\hat{\omega})$$

for every $\hat{\omega}$. So the incentive constraint (iv) is automatically satisfied. ■

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