POVERTY AND SELF-CONTROL

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"When you ain't got nothin', you got nothin' to lose." *Bob Dylan*

1. Introduction

The absence of self-control is often cited as an important contributory cause of persistent poverty, particularly (but not exclusively) in developing countries. Recent research indicates that the poor not only borrow at high rates,[1] but also forego profitable small investments.[2] To be sure, traditional theory — based on high rates of discount and minimum subsistence needs — can take us part of the way to an explanation. But it cannot provide a full explanation, for the simple reason that the poor exhibit a documented desire for commitment.[3] The fact that individuals are often willing to pay for commitment

[1]Informal interest rates in developing countries are notoriously high; see, for example Aleem (1990). But even formal interest rates are extremely high; for instance, the rates charged by microfinance organizations. Bangladesh recently capped formal microfinance interest rates at 27% per annum, a restriction frowned upon by the *Economist* (2010). Banerjee and Mullainathan (2010) cite other literature and argue that such loans are taken routinely and not on an emergency basis.

[2]Goldstein and Udry (1999) and Udry and Anagol (2006) document high returns to agricultural investment in Ghana, even on small plots, while Duflo, Kremer, and Robinson (2010) identify high rates of return to small amounts of fertliizer use in Kenya, and de Mel, McKenzie, and Woodruff (2008) demonstrate high returns to microenterprise in Sri Lanka. Banerjee and Duflo (2005) cite other studies that also show high rates of return to small investments.

[3]See, for example, Shipton (1992) on the use of lockboxes in Gambia, Benartzi and Thaler (2004) on employee commitments to save out of future wage increases in the United States, and Ashraf, Karlan, and Yin (2006) on the use of a commitment savings product in the Philippines. Aliber (2001), Gugerty (2001, 2007) and Anderson and Baland (2002) view ROSCA participation as a commitment device; see also the theoretical contributions of Ambec and Treich (2007) and Basu (2011). Duflo, Kremer, and Robinson

devices, such as illiquid deposit accounts, suggests that time inconsistency and imperfect self-control are important explanations for low saving and high borrowing, complementary to those based on impatience, minimum subsistence or a failure of aspirations.

A growing literature already recognizes that the (in)ability to exercise self-control is central to the study of intertemporal behavior.[4] Our interest lies in how self-control and economic circumstances *interact*. If self-control (or the lack thereof) is a fixed trait, independent of personal economic circumstances, then the outlook for policy interventions that encourage the poor to invest in their futures – particularly one-time or short-term interventions – is not good. But another possibility merits consideration: poverty *per se* may damage self-control. If that hypothesis proves correct, then the chain of causality is circular, and poverty is itself responsible for the low self-control that perpetuates poverty.[5] In that case, policies that help the poor *begin* to accumulate assets may be highly effective, even if they are temporary.

The preceding discussion motivates the central question of this paper: is there some *a priori* reason to expect poverty to perpetuate itself by undermining an individual's ability to exercise self-control? Our objective requires us to define self-control formally and precisely. The term itself implies an internal mechanism, so we seek a definition that does not reference any externally-enforced commitment devices. Following Strotz (1956), Phelps and Pollak (1968) and others, we adopt the view that self-control problems arise from time-inconsistent preferences: the absence of self-control is on display when an individual is unable to follow through on a desired plan of action. What then constitutes the *exercise* of self-control? We take guidance from the seminal work of the psychologist George Ainslee (1975, 1992), who argued that people maintain personal discipline by adopting private rules (e.g., "never eat dessert"), and then construing local deviations from a rule as having global significance (e.g., "if I eat dessert today, then I

(2010) explain fertilizer use (or the lack of it) in Kenya as a lack of commitment. In the ongoing debate on whether to overhaul the public distribution system for food in India to an entirely cash-based program, individual commitment issues figure prominently; see Khera (2011).

[4]See, for instance, Akerlof (1991), Ainslie (1992), Thaler (1992), Laibson (1997), or O'Donoghue and Rabin (1999). There are social aspects to the problem as well. Excess spending may be generated by discordance within the household (e.g., husband and wife have different discount factors) or by demands from the wider community (e.g., sharing among kin or community).

[5]Arguments based on aspiration failures generate parallel traps: poverty can be responsible for frustrated aspirations, which stifle the incentive to invest. See, e.g., Appadurai (2004), Ray (2006), Genicot and Ray (2009) and the recent *United Nations Development Program* Report for Latin America based on this methodology. However, this complementary approach does not generate a demand for commitment devices.

will probably eat dessert in the future as well"). It is natural to model such behavior as a subgame-perfect Nash equilibrium of a dynamic game played by successive incarnations of the single decision-maker.[6] For such a game, any equilibrium path is naturally interpreted as a personal rule, in that it describes the way in which the individual is supposed to behave. Moreover, history-dependent equilibria can capture Ainslee's notion that local deviations from a personal rule can have global consequences.[7] For example, in an intrapersonal equilibrium, an individual might correctly anticipate that violating the dictum to "never eat dessert" will trigger an undesirable behavioral pattern. Under that interpretation, the scope for exercising self-control is sharply defined by the set of outcomes that can arise in subgame-perfect Nash equilibria.

We assume that time-inconsistency arises from *quasi-hyperbolic discounting* (also known as $\beta\delta$-discounting), a standard model of intertemporal preferences popularized by Laibson (1994, 1996, 1997) and O'Donoghue and Rabin (1999). To determine the full scope for self-control, we study the set of *all* subgame-perfect Nash equilibria. To avoid excluding any viable personal rules, we impose no restrictions whatsoever on strategies (we do not require stationarity, for instance, or that the decision-maker punish deviations by reverting to the Markov-perfect equilibrium). This approach contrasts with the vast majority of the existing literature, which focuses almost exclusively on Markov-perfect equilibria (which allow only for payoff-relevant state-dependence), thereby ruling out virtually all interesting personal rules.[8] By studying the entire class of subgame-perfect Nash equilibria, we can determine when an individual can exercise sufficient self-control to accumulate greater wealth, and when she cannot. In particular, we can ask whether self-control is more difficult when initial assets are low, compared to when they are high.

The model we use is standard. There is a single asset which can be accumulated or depleted at some fixed rate of return. By using suitably defined present values, all flow incomes are nested into the asset itself. The core restriction is a strictly positive lower bound on assets, to be interpreted as a credit constraint. In other words, the individual cannot instantly consume *all* future income. The lower bound may be interpreted as referring to that fraction of present-value income which she cannot currently consume.

[6] This approach is originally due to Strotz (1956).

[7] This interpretation of self-control has been offered previously by Laibson (1997), Bernheim, Ray, and Yeltekin (1999), and Benhabib and Bisin (2001). See Bénabou and Tirole (2004) for a somewhat different interpretation of Ainslee's theory.

[8] Exceptions include Laibson (1994), Bernheim, Ray, and Yeltekin (1999), and Benhabib and Bisin (2001).

Apart from this lower bound, the model is constructed to be scale-neutral. We assume that individual payoff functions are homothetic, so we deliberately eliminate any preconceived relationship between assets and savings that is dependent on preferences alone. (We return to this point when connecting our model to related literature.) Discounting is quasi-hyperbolic.

It is notoriously difficult to characterize the set of subgame-perfect Nash equilibria (or equilibrium values) for all but the simplest dynamic games, and the problem of self-control we study here is, alas, no exception. We therefore initially examined our central question by solving the model numerically using standard tools. Figure 1 illustrates the results of one such exercise (which we explain at greater length later in the paper). The horizontal axis measures assets in the current period, and the vertical axis measures assets in the next period. Thus, points above, on, and below the 45 degree line indicate asset accumulation, maintenance, and decumulation respectively. In this exercise, there is an asset threshold below which *all* equilibria lead to decumulation. Thus, with low assets, it is impossible to accumulate assets by exercising self-control through any viable personal rule; on the contrary, assets necessarily decline until the individual's liquidity constraint binds. In short, we have a poverty trap.

However, above that threshold, there are indeed viable personal rules that allow the individual to accumulate greater assets. Moreover, as we will see later, the most attractive equilibria starting from above the critical threshold lead to unbounded accumulation.

The example suggests both our central conjecture and a (deceptively) simple intuitive argument in support of it. If imperfect capital markets impose limits on the extent to which an individual can borrow against future income, then potential intrapersonal "punishments" (that is, the consequences of deviating from a personal rule) cannot be all that bad when assets are low. If these mild repercussions are suitably anticipated, an individual will fail to exercise self-control. However, when an individual has substantial assets, she also has more to lose from undisciplined future behavior, and hence potential punishments are considerably more severe (in relative terms). So an individual would be better able to accumulate additional assets through the exercise of self-control when initial assets are higher. Obviously, if time inconsistency is sufficiently severe, decumulation will be unavoidable regardless of initial assets, and if it is sufficiently mild, accumulation will be possible regardless of initial assets (provided the individual is sufficiently patient). But for intermediate degrees of time inconsistency, we would expect

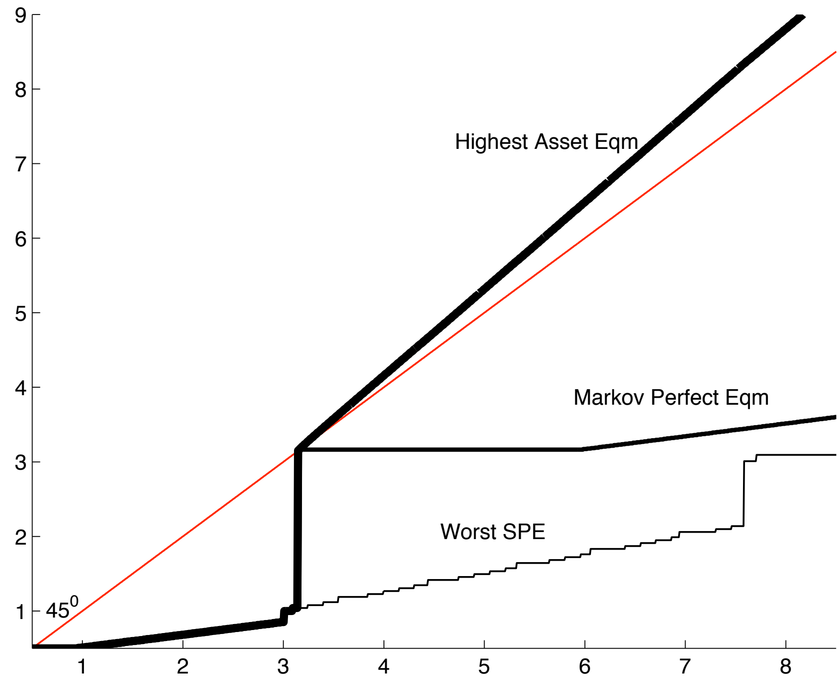


FIGURE 1. ACCUMULATION AT DIFFERENT ASSET LEVELS.

decumulation to be unavoidable with low assets, and accumulation to be feasible with high assets.

It turns out, however, that the problem is considerably more complicated than this simple intuition suggests. (The overwhelmingly numerical nature of our earlier working paper, Bernheim, Ray, and Yeltekin (1999), bears witness to this assertion.) The credit constraint at low asset levels infects individual behavior at *all* asset levels. In particular, they affect the structure of "worst personal punishments" in complex ways, as assets are scaled up. The example of Figure 1 illustrates this point quite dramatically: there are asset levels at which the *lowest* level of continuation assets jumps up discontinuously. As assets cross those thresholds, the worst punishment becomes *less* rather than *more* severe, contrary to the intuition given above. Thus, on further reflection, it is not at all clear that the patterns exhibited in Figure 1 will emerge more generally.

Our main theoretical result demonstrates, nevertheless, that the central qualitative properties of Figure 1 do persist. For intermediate degrees of time inconsistency (such that accumulation is feasible from some but not all asset levels), there is a threshold asset level below which accumulation is impossible, and above which decumulation is avoidable. There is always an asset level below which liquid wealth is exhausted in finite time (and hence a poverty trap), as well as a level above which the most attractive equilibria give rise to unbounded accumulation.

One might object to our practice of examining the entire set of subgame-perfect equilibria on the grounds that many such equilibria may be unreasonably complex. On the contrary, we show that worst punishments have a surprisingly simple "stick-and-carrot" structure:[9] following any deviation from a personal rule, the individual consumes aggressively for one period, and then returns to an equilibrium path that maximizes her (equilibrium) payoff *exclusive* of the hyperbolic factor. Thus, all viable personal rules can be sustained without resorting to complex forms of history-dependence.[10]

Our analysis has a number of provocative implications for economic behavior and public policy. First (and most obviously), the relationship between assets and self-control argues for the use of "pump-priming" interventions that encourage the poor to start saving, and rely on self-control to sustain frugality at higher levels of assets. Second, policies that improve access to credit (thereby relaxing liquidity constraints) reduce the level of assets at which asset accumulation becomes feasible, thereby helping more individuals to become savers (although those who fail to make the transition fall further into debt). Intuitively, with greater access to credit, the consequences of a break in discipline become more severe, and hence that discipline is easier to sustain to begin with. Third, the opportunities to make commitments may be significantly less valuable to those with self-control problems than previous analyses have implied. For example, in certain circumstances, individuals with self-control problems will avoid opportunities to lock up funds (e.g., in retirement accounts or fixed deposit schemes), even when they wish to save. This occurs when desired saving exceeds the maximum amount that can be locked up, but not by too much. In such cases, locking up funds moderates the consequences of a lapse in discipline, thereby making self-control more difficult to sustain. Finally,

[9]Though there is a resemblance to the stick-and-carrot punishments in Abreu (1988), the formal structure of the models and the arguments differ considerably. Most obviously, Abreu considered simultaneous-move repeated games, rather than sequential-move dynamic games with state variables.

[10]Indeed, Markov equilibria in this model appear to be more "complex", despite their "simple" dependence on just the payoff-relevant state. They typically involve several jump discontinuities, and their payoff behavior as a function of initial assets, suitably normalized, is often nonmonotonic. Also, identifying Markov equilibria is more computationally challenging than determining the key features of subgame-perfect equilibria.

we argue that the model can potentially provide an explanation for the observed "excess sensitivity" of consumption to income.

As noted above, we build on an earlier unpublished working paper (Bernheim, Ray, and Yeltekin (1999)), which made our main points through simulations, but did not contain theoretical results. Our work is most closely related to that of Banerjee and Mullainathan (2010), who also argue that self-control problems give rise to low asset traps. Though the object of their investigation is similar, their analysis has little in common with ours. They examine a novel model of time-inconsistent preferences, in which rates of discount differ from one good to another. "Temptation goods" (those to which greater discount rates are applied) are inferior by assumption; this assumed non-hometheticity of preferences automatically builds in a tendency to dissave when resources are limited, and to save when resources are high.

It is certainly of interest to study poverty traps by hardwiring non-homothetic self-control problems into the structure of preferences. Whether a poor person spends proportionately more on temptation goods than a rich person (alcohol versus iPads, say) then becomes an empirical matter. But we avoid such hardwiring entirely by studying homothetic preferences in an established model of time-inconsistency. The phenomena we study are traceable to a single built-in asymmetry: an imperfect credit market. Every scale effect in our setting arises from the interplay between credit constraints and the incentive compatibility constraints for personal rules. The resulting structure, in our view, is compelling in that it requires no assumption concerning preferences that must obviously await further empirical confirmation. In summary, though both theories of poverty traps invoke self-control problems, they are essentially orthogonal (and hence potentially complementary): Banerjee and Mullainathan's analysis is driven by assumed scaling effects in rewards, while ours is driven by scaling effects in punishments arising from assumed credit market imperfections.[11]

The rest of the paper is organized as follows. Section 2 presents the model, and Section 3 characterizes the set equilibrium values. Section 4 defines self control, and Section 5 studies the relationship between self-control and the initial level of wealth. Section

[11]Our model is also related to Laibson (1994) and Benhabib and Bisin (2001), except for the all-important difference of an imperfect credit market. These two papers consider history-dependent strategies in a *fully* scalable model, in which both preferences are homothetic and there is no credit constraint. It follows, as we observe more formally below, that every equilibrium path can be replicated, by scaling, at all levels of initial assets, so that there is no relationship between poverty and self-control.

6 describes some implications of the theory. Section 7 conducts numerical exercises to supplement the analytical findings. Section 8 presents conclusions and some directions for future research. Proofs are collected in Section 9.

2. Model

2.1. **Feasible Set and Preferences.** The feasible set links current assets, current consumption and future assets, starting from an initial asset level A_0:

$$c_t = A_t - (A_{t+1}/\alpha) \geq 0, \tag{1}$$

and, in addition, imposes a lower bound on assets

$$A_t \geq B > 0. \tag{2}$$

Our leading interpretation of the lower bound B is that it is a credit constraint.[12] For instance, if F_t stands for *financial* wealth at date t and y for income at each date, then A_t is the present value of financial and labor assets:

$$A_t = F_t + \frac{\alpha y}{\alpha - 1}.$$

If credit markets are perfect, the individual will have all of A_t at hand today, and $B = 0$. We are not directly interested in this case (our analysis presumes $B > 0$) but it is easy enough to analyze; see Laibson (1994). On the other hand, if she can borrow only some fraction $(1 - \lambda)$ of lifetime income, then $B = \lambda\alpha y/(\alpha - 1)$.

Individual have quasi-hyperbolic preferences: lifetime utility is given by

$$u(c_0) + \beta \sum_{t=1}^{\infty} \delta^t u(c_t),$$

where $\beta \in (0, 1)$ and $\delta \in (0, 1)$. We assume throughout that u has the constant-elasticity form

$$u(c) = \frac{c^{1-\sigma}}{1 - \sigma}$$

for $\sigma > 0$, with the understanding that $\sigma = 1$ refers to the logarithmic case $u(c) = \ln c$.

There is a good reason for the use of the constant-elasticity formulation. We wish our problem to be entirely scale-neutral in the absence of the credit constraint, so as to

[12]Another interpretation of B is that it is an investment in some fixed illiquid asset. We return to this interpretation when we talk about policy implications.

isolate fully the effect of that constraint. While we don't formally analyze the case in which $B = 0$, it is obvious that scale-neutrality is achieved there: any path with perfect credit markets can be freely scaled up or down with no disturbance to its equilibrium properties. Put another way, every scale effect in this paper will arise from the interplay between credit constraints and the incentive compatibility constraints for personal rules.

2.2. **Restrictions on the Model.** The *Ramsey program* from A is the asset sequence $\{A_t\}$ that maximizes

$$\sum_{t=0}^{\infty} \delta^t \frac{c_t^{1-\sigma}}{1-\sigma},$$

with initial stock $A_0 = A$. It is constructed without reference to the hyperbolic factor β.

The Ramsey program is well-defined provided that utilities do not diverge, for which we assume throughout that

$$\gamma \equiv \delta^{1/\sigma} \alpha^{(1-\sigma)/\sigma} < 1. \tag{3}$$

We will also be interested in situations in which the Ramsey program exhibits growth, which imposes

$$\delta\alpha > 1. \tag{4}$$

Under (3) and (4), it is easy to see that along the Ramsey program,

$$c_t = (1-\gamma)A_t,$$

assets grow exponentially:

$$A_{t+1} = A_0 \left(\delta^{1/\sigma}\alpha^{1/\sigma}\right)^t = A_0 \left(\gamma\alpha\right)^t,$$

and the value of the program from A — call it $R(A)$ — is finite.

Before we turn to a precise definition of equilibrium, note that when $\sigma \geq 1$, utility is unbounded below and it is possible to sustain all sorts of outcomes by taking recourse to punishments that either impose zero consumption or a progressively more punitive sequence of non-zero consumptions (see Laibson (1994) for a discussion of this point). These punishments rely on the imposition of unboundedly negative utility. We find such actions unrealistic, and eliminate them by assuming that consumption is bounded below at every asset level. More precisely, we assume that at every date,

$$c_t \geq \nu A_t, \tag{5}$$

where ν is to be thought of as a small but positive number. It is formally enough to presume that $\nu < 1 - \gamma$, so that Ramsey accumulation can occur unhindered, but the reader is free to think of this bound as tiny.

2.3. **Equilibrium.** A choice of continuation asset A' is *feasible* given A, if $A' \in [B, \alpha(1-\upsilon)A]$. A *path* is any sequence of assets with A_{t+1} feasible given A_t; so (1), (2) and (5) are satisfied. A *history* h_t at date t is a "truncated path" of assets $(A_0(h_t) \dots A_t(h_t))$ up to date t, so that $A_t \equiv A(h_t)$ is the asset level at the start of date t. A *policy* ϕ specifies a continuation asset $\phi(h_t)$ following every history, which must be feasible given $A(h_t)$. If h_t is a history and x a feasible asset choice, denote by $h_t.x$ the subsequent history generated by this choice. A policy ϕ defines a *value* V_ϕ by

$$V_\phi(h_t) \equiv \sum_{s=t}^{\infty} \delta^{s-t} u\left(A(h_s) - \frac{\phi(h_s)}{\alpha}\right),$$

where h_s (for $s > t$) is recursively defined from h_t by $h_{s+1} = h_s.\phi(h_s)$ for $s \geq t$. Similarly, ϕ also defines a *payoff* P_ϕ by

$$P_\phi(h_t) \equiv u\left(A(h_t) - \frac{\phi(h_t)}{\alpha}\right) + \beta\delta V_\phi(h_t.\phi(h_t)),$$

for every history h_t. Values exclude the hyperbolic factor β, while payoffs include them.

An *equilibrium* is a policy such that at every history h_t and x feasible given $A(h_t)$,

$$P_\phi(h_t) \geq u\left(A(h_t) - \frac{x}{\alpha}\right) + \delta\beta V_\phi(h_t.x). \tag{6}$$

Equation (6) makes it clear that an equilibrium may be viewed as the assignment of a continuation value for every choice of continuation asset (at any given history), and then taking the *actual* continuation asset at that history to be the one that maximizes the right hand side of (6) over all these specifications. For some of our observations, it will be useful to presume that a convex set of equilibrium continuation values is available at every asset level. For this reason, we suppose that continuation values can be chosen (if needed) using a public randomization device.

3. Existence and Characterization of Equilibrium

For each initial asset level $A \geq B$, let $\mathcal{V}(A)$ be the set of all equilibrium values available at asset level A. If $\mathcal{V}(A)$ is nonempty, let $H(A)$ and $L(A)$ be its supremum and infimum

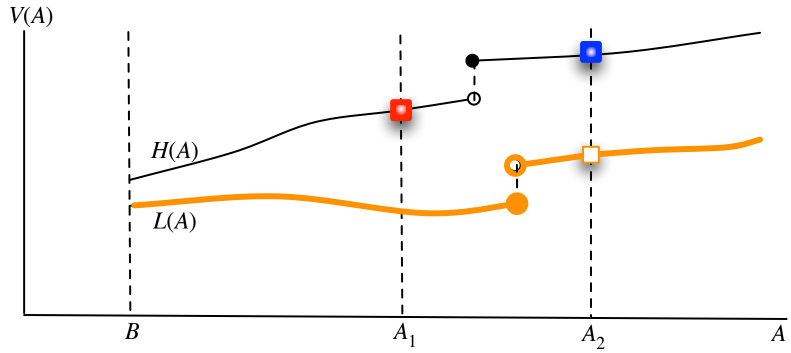


FIGURE 2. EQUILIBRIUM VALUES.

values. It is obvious from our assumed lower bound on consumption and from utility convergence (see (3)) that

$$-\infty < L(A) \leq H(A) \leq R(A) < \infty,$$

where $R(A)$ is the Ramsey value defined earlier. Once we rule out unrealistic cascades of punishments that generate arbitrarily negative utility, a tighter and more intuitive bound is available for worst punishments:

OBSERVATION **1.** *Suppose that $\mathcal{V}(A)$ is nonempty for every $A \geq B$. Then*

(7) $$L(A) \geq u\left(A - \frac{B}{\alpha}\right) + \delta L(B) \geq u\left(A - \frac{B}{\alpha}\right) + \frac{\delta}{1-\delta} u\left(\frac{\alpha - 1}{\alpha} B\right)$$

Notice how Observation 1 kicks in as long as we place *any* (small) lower bound on consumption, as described in (5).[13] It gives us an anchor to iterate a self-generation map, both for analytical use and for equilibrium computation. To this end, consider a nonempty-valued correspondence $\mathcal{W}$ on $[B, \infty)$ such that for all $A \geq B$,

(8) $$\mathcal{W}(A) \subseteq \left[u\left(A - \frac{B}{\alpha}\right) + \frac{\delta}{1-\delta} u\left(\frac{\alpha - 1}{\alpha} B\right), R(A)\right].$$

Say that $\mathcal{W}$ *generates* the correspondence $\mathcal{W}'$ if for every $A \geq B$, $\mathcal{W}'(A)$ is the collection of *all* values W such there is a feasible asset choice x and $V \in \mathcal{W}(x)$ — a

[13] In contrast, if there is no lower bound, and utilities are unbounded below, we can generate artificially low punishments by effectively using "Ponzi threats".

continuation $\{x, V\}$ in short — with

$$W = u\left(A - \frac{x}{\alpha}\right) + \delta V, \tag{9}$$

while for every feasible x', there is $V' \in \mathcal{W}(x')$ such that

$$u\left(A - \frac{x}{\alpha}\right) + \beta\delta V \geq u\left(A - \frac{x'}{\alpha}\right) + \beta\delta V'. \tag{10}$$

Given Observation 1 and the Ramsey upper bound on equilibrium values, standard arguments tell us that the equilibrium correspondence $\mathcal{V}$ generates itself, and indeed, it contains any other correspondence that does so.

Define a sequence of correspondences on $[B, \infty)$, $\{\mathcal{V}_k\}$, by

$$\mathcal{V}_0(A) = \left[u\left(A - \frac{B}{\alpha}\right) + \frac{\delta}{1-\delta}u\left(\frac{\alpha - 1}{\alpha}B\right), R(A)\right].$$

for every $A \geq B$, and recursively, $\mathcal{V}_k$ generates $\mathcal{V}_{k+1}$ for all $k \geq 0$. It is obvious that the graph of $\mathcal{V}_k$ contains the graph of $\mathcal{V}_{k+1}$. We assert

PROPOSITION **1.** *An equilibrium exists from any initial asset level, so that the equilibrium correspondence $\mathcal{V}$ is nonempty-valued. Moreover, for every $A \geq B$,*

$$\mathcal{V}(A) = \bigcap_{k=0}^{\infty} \mathcal{V}_k(A). \tag{11}$$

and $\mathcal{V}$ has closed graph.

This proposition is useful in that it establishes existence of equilibrium, though the method used may not apply more generally to all games with state variables.[14] The "generation logic" of the proposition inspires algorithms for numerical calculations along well-known lines, which we use in Section 7.[15]

Figure 2 illustrates a set of equilibrium values. Imagine supporting the highest possible value $H(A_1)$ at asset level A_1. That might require the choice of asset A_2 — and so would be the stipulation of the equilibrium policy — followed by the continuation value $H(A_2)$. Any other choice would be followed by other continuation values designed to discourage that choice, so that the inequality in (6) holds. The figure illustrates the "best"

[14]For more general existence theorems, see Goldman (1980) and Harris (1985).

[15]Incidentally note that the closed-graph property does not follow from a standard nested compact sets argument, because the sets in question (the graphs of $\mathcal{V}_k$) are not compact.

way of doing this under the presumption that the equilibrium value set is compact-valued and has closed graph: simply choose the worst continuation value $L(x)$ if $x \neq A_2$.

4. SELF CONTROL

Viewed in the spirit of Ainslee's definition, self-control refers to a *possibility*; that is, to a feature of *some* element of the equilibrium correspondence. One might ask, for instance, if the Ramsey outcome itself is an equilibrium. That would require, of course, that the agent entirely transcend her hyperbolic urges. All other attempts, including accumulation at rates close to the Ramsey path, must then be deemed a failure (of self-control), which we find too strong. We therefore impose a weak definition: there is *self-control* at asset level A if the agent is capable of positive saving at A in *some* equilibrium.

To be sure, we might be interested in whether the individual is capable of indefinite accumulation. Say that there is *strong self-control* at A if the agent is capable of unbounded accumulation — i.e., $A_t \to \infty$ — for some equilibrium path emanating from A.

Now we look at the flip side of self-control. Clearly, we must define the absence of self-control as a situation in which accumulation isn't possible under *any* equilibrium. But that failure is compatible with several outcomes: the stationarity of assets, a downward spiral of assets to a lower level that nevertheless exceeds the lower bound, or a progressive downward slide all the way to the minimal level B. In a symmetric way, we single out two features: say that *self control fails* at A if every equilibrium continuation asset is strictly smaller than A, and more forcefully, that there is a *poverty trap at asset* A if in every equilibrium, assets decline over time from A to the lower bound B.

There is intermediate ground between strong self-control and a poverty trap: it is, in principle, possible for an agent to be incapable of indefinite accumulation, while at the same time she can avoid the poverty trap.

That said, there are situations in which self-control is possible at *all* asset levels. For instance, if β is close to 1, there is (almost) no time-inconsistency and all equilibria should exhibit accumulation, given our assumption that the Ramsey program involves indefinite growth. Conversely, if the agent exhibits a high degree of hyperbolicity (β small), there may be a failure of self-control no matter what asset level we consider. Call a case *uniform* if there are no switches: either there is no failure of self control at every asset level, or there is no self-control at every asset level.

A good example of uniformity is given by the case in which credit markets are perfect. While we don't study perfect credit markets in this paper, the observation is worth noting: if continuation asset x can be sustained at asset level A, then continuation asset λx can be just as easily sustained when the asset level is λA, for any $\lambda > 0$. Indeed, we've deliberately constructed the model in this fashion, so to understand better the "direction of scale bias" created by introducing imperfect credit markets.

The *nonuniform* cases, then, are of interest to us. In these cases, self-control is possible at some asset level A, while there is a failure of self-control at some other asset level A'. Whether A' is larger or smaller than A, or indeed, whether there could be several switches back and forth, are among the central issues that we wish to explore. It should be added that while we do not have a full characterization of when a case is nonuniform, such cases exist in abundance (we confirm this by numerical analysis).

We close this section with an intuitive yet nontrivial characterization of self control. Consider the *largest continuation asset*: the highest value of equilibrium asset $X(A)$ sustainable at A. The closed-graph property of Proposition 1 guarantees that $X(A)$ is well-defined and usc, and a familiar single-crossing argument tells us that it is nondecreasing. Note that $X(A)$ isn't necessarily the value-maximizing choice of asset; it could well be higher than that. Yet $X(A)$ is akin to a sufficient statistic that can be used to characterize all the concepts in this section.

PROPOSITION **2.** (i) *Self control is possible at A if and only if $X(A) > A$.*

(ii) *Strong self control is possible at A if and only if $X(A') > A'$ for all $A' \geq A$.*

(iii) *There is a poverty trap at A if and only if $X(A') < A'$ for all $A' \in (B, A]$.*

(iv) *There is uniformity if and only if $X(A) \geq A$ for all $A \geq B$, or $X(A) \leq A$ for all $A \geq B$.*

Parts (i) and (iv) of the proposition are obvious, but parts (ii) and (iii), while intuitive, need a more extensive argument. Part (iii) will indeed follow from the additional observation that X is nondecreasing and usc. Part (ii) will need more work to prove. Yet, it is useful to take the proposition on faith for now, as it will help us in visualizing the proof of the main theorem.

5. Initial Assets and Self-Control

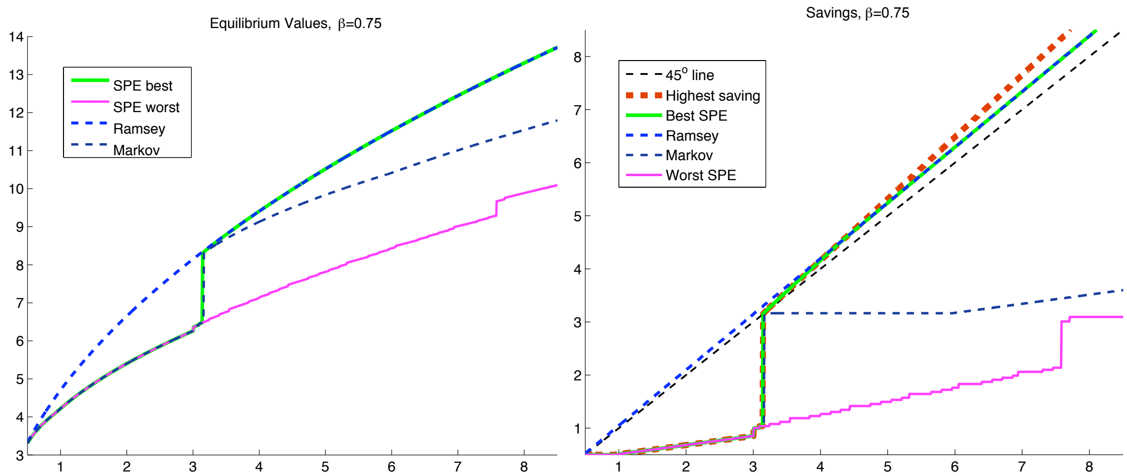
It is obvious that if $B > 0$, then "scale-neutrality" fails. For instance, at asset level B, it isn't possible to decumulate assets (by assumption), while that may be an equilibrium outcome at $A > B$. This rather simplistic failure of neutrality opens the door to all sorts of more interesting failures. For instance, accumulation at some asset level A may be sustained by the threat of decumulation in the event of non-compliance; such threats will not be credible at asset levels close to B.

These internal checks and balances are not merely technical, but descriptive (we feel) of individual ways of coping with commitment problems. One coping mechanism is "external": an individual might commit to a fixed deposit account if available, or even accounts that force her to make regular savings deposits in addition to imposing restrictions on withdrawal. We will have more to say about such mechanisms below. But the other coping mechanism is "internal": an agent might react to an impetuous expenditure on her part by engaging in a behavior shift; for instance, she might go on a temporary consumption spree. In our theory, such a binge must be a valid continuation equilibrium. The threat of a "credible binge" might then help to keep the agent in check.

With this "internal mechanism" in mind, let's ask why an abundance of assets might help an individual to exhibit self-control. The ability to exercise control must depend on the severity of the consequences following an impetuous act of consumption. One simple intuition is that those consequences are more severe when the individual has more assets, and hence more to lose. But we know that such an argument can run either way.[16]

In addition, the "severity of punishment" (even if suitably normalized) isn't monotonic in assets. Figure 3 shows a numerical example which makes this point. The left panel shows various value selections from the equilibrium correspondence, as also the Ramsey value. The lowest selection is $L(A)$. It jumps several times; the diagram shows one such jump between the values 7 and 8. The right hand panel describes the corresponding choice of continuation assets. The jump in $L(A)$ shows that in general, punishment values (even after deflating by higher asset values) cannot be decreasing in A.

[16]For instance, in moral hazard problems with limited liability, a poor agent might face more serious incentive problems than a rich one; see, e.g., Mookherjee (1997). On the other hand, the curvature of the utility function will permit the inflicting of higher *utility* losses on poorer individuals, alleviating moral hazard and conceivably permitting the poor to be better managers (Banerjee and Newman (1991)).



(A) Values (B) Policies

FIGURE 3. JUMPS IN VALUES AND POLICIES

The jump in L is related to the failure of lower hemicontinuity of the constraint set in the implicit minimization problem that defines lowest values. That constraint set is constructed from the graph of the equilibrium value correspondence, in which all continuation values must lie. As assets converge down to some limit, discontinuously lower values may become available, and as the numerical example illustrates, this phenomenon cannot be ruled out in general. We return to this point after we explain the simple structure of worst punishments in this model.

5.1. **Worst Punishments.** We will show that worst punishments involve a single spell of "excessive" expenditure, followed by a return to (approximately) the best possible continuation value. To formalize this notion, define, for any $A > B$, $H^{-}(A)$ by the left limit of $H(A^n)$ as A^n converges up to A, with $A^n < A$ for all n. This is a well-defined concept because H is nondecreasing and therefore possesses limits from the left.

PROPOSITION **3.** *The worst equilibrium value at any asset level A is implemented by choosing the smallest possible continuation asset at A; call it Y. Moreover, if $Y > B$, the associated continuation value V satisfies*

$$V \geq H^{-}(Y).$$

The proof is simple and instructive enough to be included in the main text.

Proof. Let Y be the smallest equilibrium choice of continuation asset at A, with associated continuation value V. Of course,

$$u\left(A - \frac{Y)}{\alpha}\right) + \beta\delta V \geq D(A), \tag{12}$$

where $D(A)$ is the supremum of all "deviation payoffs" where every deviation to an alternative asset choice is "punished" by the lowest equilibrium value available at that asset.[17] If (12) is slack, it is easy to show that Y must equal B and that V can be set equal to $L(B)$.[18] That generates the lowest possible equilibrium value at A and there is nothing left to prove; see the first inequality in Observation 1.

Otherwise (12) is binding for Y. In this case,

$$u\left(A - \frac{Y}{\alpha}\right) + \beta\delta V = D(A) \leq u\left(A - \frac{A'}{\alpha}\right) + \beta\delta V'. \tag{13}$$

for any other equilibrium continuation $\{A', V'\}$ at A. Because $A' \geq Y$ by assumption, (13) shows that $V' \geq V$. It follows that

$$u\left(A - \frac{Y}{\alpha}\right) + \delta V \leq u\left(A - \frac{A'}{\alpha}\right) + \delta V', \tag{14}$$

so that once again, $\{Y, V\}$ implements minimum value at A.

To complete the proof, suppose that $Y > B$ while at the same time, $V < H^{-}(Y)$. Then it is obviously possible to reduce Y slightly while increasing continuation value at the same time.[19] Moreover, the new continuation has higher payoff, so it must be supportable as an equilibrium. Yet it has a lower continuation asset, which contradicts the definition of Y. ■

The heart of the argument resides in (14). If two continuations generate the same payoff, the continuation that exhibits the larger upfront consumption must have the lower value. Payoffs include the factor β, which values present consumption. When β is "removed", as it is in the computation of *value*, the continuation with the larger current consumption

[17]The function $D(A)$ will be formally defined in Section 9, where we deal with various technicalities arising from lack of the continuity in the value correspondence.

[18]For details, see Footnote 34 in Section 9.

[19]Because $V < H^{-}(Y)$, there exists $Y' < Y$ and $V' \in \mathcal{V}(Y)$ such that $V' > H^{-}(Y)$.

has lower value. This is why worst punishments exhibit a large binge to begin with; in fact, the largest possible credible binge. That binge is then followed by a reversion to the best possible equilibrium value — or approximately so, in a sense made precise in the proposition.[20]

Two more remarks are worth noting about lowest values, or optimal punishments. First, the associated actions have an extremely simple and plausible structure. No unrealistically complex rules are followed that might justify a restriction to "simpler" notions, such as Markov punishments. An individual doesn't fall of the wagon forever, but there is still retribution for a deviation: a binge is followed by a further binge, the fear of which acts as a deterrent. After that, the individual is back on the wagon. Second, there is a sense in which these punishments are reasonably immune to renegotiation. While the earlier, deviating self fears the low-value path, the self that inflicts the punishment is actually treated rather well: he gets to enjoy a free binge, followed by the promise of self-control being exercised in the future.

Finally, while optimal punishments are reminiscent of the carrot-and-stick property for optimal penal codes in repeated games (Abreu (1988)), there is no reason why that property should hold, in general, for games with state variables, of which our model is an example. In this model, the particular structure arises from the hyperbolic factor β. That parameter dictates that the most effective punishments are achieved by as much excess consumption "as possible" in the very first period of the punishment. From the point of view of the deviator, that first period lies in *his* future, and as such it is a bad prospect (hence an effective punishment). From the point of view of the punisher, the punishment might actually yield pleasing equilibrium payoffs. That is, the carrot-and-stick feature is very much in the eyes of the deviator, and not in the eye of the punisher, a distinction that is often not present in repeated games.

5.2. **The Relationship Between Wealth and Self-Control.** Given Proposition 3, we can generate a bit more intuition on the issue of "jumps" in worst punishments. Let the continuation $\{Y, V\}$ support the lowest value at A. Let's look at the no-deviation condition (12) more closely. Neglecting some technical matters for the sake of exposition, $D(A)$ refers to the payoff obtained by deviating to another continuation asset d, followed

[20]We note again that reversion to the best continuation value occurs, provided that the asset level post-binge is strictly higher than B, and provided that the best value selection is continuous at that asset level. Otherwise the return is not necessarily to the best equilibrium continuation: recall the definition of H^-.

thereafter by the worst punishment starting from d with value $L(d)$. In general, $d > Y$ for the reasons outlined in Proposition 3. If the no-deviation condition is binding,

$$u\left(A - \frac{Y)}{\alpha}\right) + \beta\delta V = u\left(A - \frac{d)}{\alpha}\right) + \beta\delta L(d).$$

Now increase A. Because $d > Y$, the strict concavity of utility forces the right hand side of this constraint to increase more quickly than the left, holding constant both the earlier equilibrium choice Y and the deviation d. That places pressure on the incentive constraint. Indeed, depending on the shape of the equilibrium value correspondence, both Y and V may have to be changed discretely, leading to an upward jump in L.[21]

The possibility that worst equilibrium values can abruptly rise with wealth leads to the nihilistic suspicion that no general connection can be made between wealth and self-control. Nevertheless, not one of the extensive numerical examples that we have studied bears out this suspicion. Bernheim, Ray and Yeltekin (1999) show that either we are in one of the two uniform cases (accumulation possible everywhere, or accumulation impossible anywhere), or the situation looks quite generically like Figure 3. Initially, there is asset decumulation in every equilibrium, followed by the crossing of a threshold at which indefinite accumulation becomes possible. The non-uniform cases invariably display a failure of self-control to begin with (at low asset levels), followed by the emergence and maintenance of self-control after a certain asset threshold has been crossed.

The main proposition of this paper supports the numerical analysis:

PROPOSITION **4.** *In any non-uniform case:*

(i) *There is* $A_1 > B$ *such that every* $A \in [B, A_1)$ *exhibits a poverty trap.*

(ii) *There is* $A_2 \geq A_1$ *such that every* $A \geq A_2$ *exhibits strong self-control.*

The proposition states that in any situation where imperfect credit markets are sufficient to disrupt uniformity, the lack of scale neutrality manifests itself in a particular way. At low enough wealth levels, individuals are unable to exert self-control, and their actions must generate a poverty trap. At high enough wealth levels, indefinite accumulation is possible. There is, of course, no reason *a priori* why this must be the case. It is possible, for instance, that there a maximal asset level beyond which accumulation ceases

[21] Essentially, the constraint set is not continuous in A, leading to a failure of the maximum theorem.

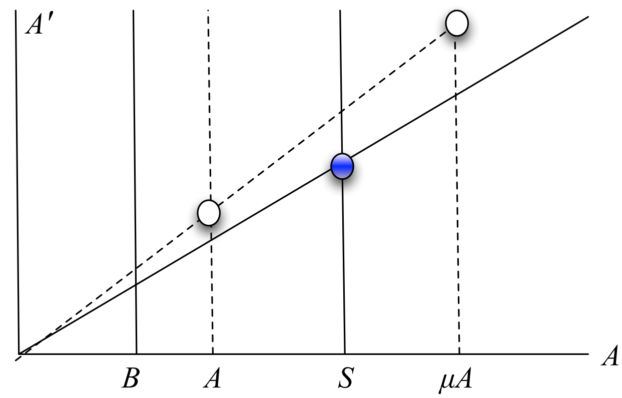


FIGURE 4. MODIFIED SCALING.

altogether, or that there are (infinitely) repeated intervals along which accumulation and decumulation occur alternately. But the proposition rules out these possibilities.

This proposition provides partial vindication for the numerical analysis conducted in Bernheim, Ray and Yeltekin (1999). One should compare this finding with the main observation in Banerjee and Mullainathan (2010). They make the same point as we do here and (numerically) in our earlier work. Self-control problems give rise to low asset traps. But the analysis is different. They study time-inconsistent preferences over multiple goods in which rates of discount differ from one good to another. "Temptation goods" have higher discount rates attached to them. They are taken to be inferior by assumption. This assumed non-hometheticity of preferences generates a tendency to dissave when resources are limited. Our non-homotheticity manifests itself not via preferences but through the imperfection of credit markets. As we've discussed, there is no *a priori* presumption regarding the direction of that non-homotheticity.

In fact, the proposition fails to establish the existence of a *unique* asset threshold beyond which there is self-control, and below which there isn't. A demonstration of this stronger result is hindered in part by the possibility that worst punishments can move in unexpected ways with the value of initial assets. In fact, we conjecture that a "single crossing" is possibly not to be had, at least under the assumptions that we have made so far. From this perspective, the fact that "ultimately" all multiple crossings must cease — which is part of the assertion in the proposition — appears surprising, and the remainder of this section is devoted to an informal exposition of the proof.

5.3. **An Informal Exposition of the Main Proposition.** As we've mentioned on several occasions, it is the presence of imperfect credit markets that destroys scale-neutrality in our model. (The constant elasticity of preferences assures us that otherwise, the situation would be fully scale-neutral.) Yet variations of scale-neutrality survive. One variation that is particularly germane to our argument is given in Observation 2 below.

To state it, define an asset level $S \geq B$ to be *sustainable* if there exists an equilibrium that permits indefinite maintenance of S. It is important to appreciate that a sustainable asset level need not permit strict accumulation, and more subtly, an asset level that permits strict accumulation *need not* be sustainable.[22]

OBSERVATION **2.** *Let $S > B$ be a sustainable asset level. Define $\mu \equiv S/B > 1$. Then for any initial asset level $A \geq B$, if continuation asset A' can be supported as an equilibrium choice, so can the continuation asset $\mu A'$ starting from μA.*

Figure 4 illustrates the Observation. First think of S as a new lower bound on assets. Then it is plain that any equilibrium action under the old lower bound B can be simply scaled up using the ratio of S to B, which is μ. Indeed, if we replaced the word "sustainable" by the phrase "physical minimum", then the Observation would be trivial. We would simply scale up *all* continuation actions from the old equilibrium specification to the new one. However, S is not a physical minimum. Deviations to asset levels below S are available, and there is no version of such a deviation in the earlier equilibrium that can be rescaled (deviations below B are not allowed, after all). Nevertheless, the formal proof of Observation 2 (see formal statement and proof as Lemma 7 in Section 9) shows that given the concavity of the utility function, such deviations can be suitably deterred. Thus, while S isn't a physical lower bound, it permits us to carry out the same scaling we would achieve if it were.

Let's use Observation 2 to establish the first part of the proposition:

(i) There is $A_1 > B$ such that every $A \in [B, A_1)$ exhibits a poverty trap.

Recall that $X(A)$ to be the largest continuation asset in the class of all equilibrium outcomes at A. By Proposition 2, we will need to show that there is an asset level $A_1 > B$ such that $X(A) < A$ for all $A \in (B, A_1)$. Suppose, now, that the proposition is false; then — relegating the impossibility of eternal wiggles back and forth to the more

[22]The continuation values created by continued accumulation might incentivize accumulation from A, while a stationary path may not create enough incentives for self-sustenance.

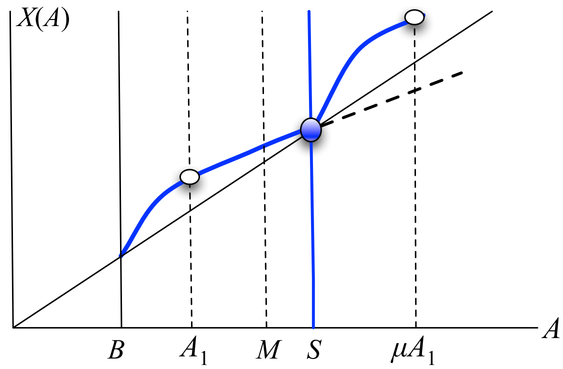


FIGURE 5. ESTABLISHING THE EXISTENCE OF A POVERTY TRAP.

formal arguments — there is $M > B$ such that $X(A) \geq A$ for all $A \in [B, M]$. Figure 5 illustrates this scenario.

Because we are in a non-uniform case, there exists an asset level A^* at which self-control fails, so by Proposition 2, $X(A^*) < A^*$. Let S be the supremum value of assets over $[B, A^*]$ for which $X(A) \geq A$. Note that *at* S, it must be the case that $X(S)$ *equals* S.[23] Because $X(S) = S$, S is sustainable, though this needs a formal argument.[24]

Now Observation 2 kicks in to assert that $X(A)$ must exceed A just to the right of S: simply scale up the value $X(A_1)$ for some A_1 close to B to the corresponding value $\mu X(A_1)$ at μA_1, where $\mu \equiv S/B$. But that is a contradiction to the way we've defined S, and shows that our initial presumption is false. Therefore $X(A) < A$ for every A close enough to B. That establishes the existence of an initial range of assets for which a poverty trap is present, and so proves (i).

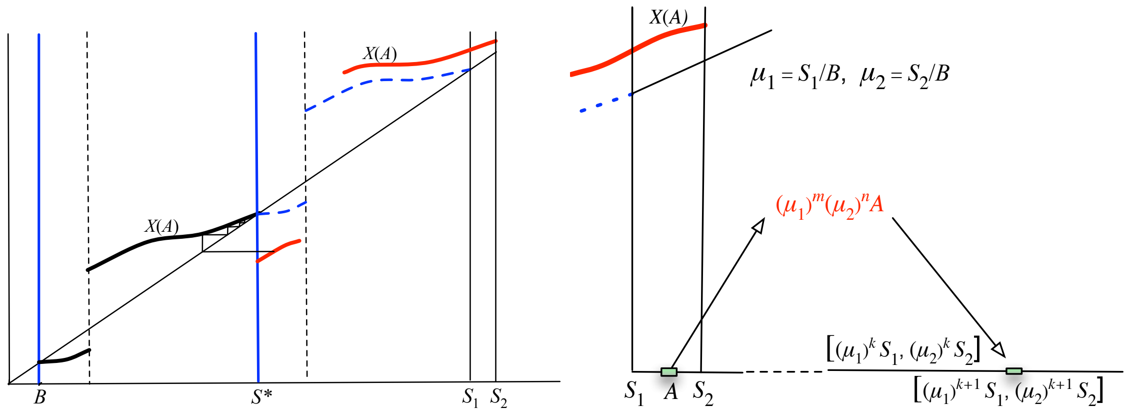
Next, we work on:

(ii) There is $A_2 \geq A_1$ such that every $A \geq A_2$ exhibits strong self-control.

By nonuniformity, there is certainly some value of A for which $X(A) > A$. If the same inequality continues to hold for all $A' > A$, then by Proposition 2, strong self-control is established, not just at A but at every asset level beyond it. So the case that we need to worry about is one in which $X(A^*) \leq A^*$ for some asset level still higher than A. See

[23]It can't be strictly lower, for then X would be jumping down at S, and it can't be strictly higher for then we could find still higher asset levels for which $X(A) \geq A$.

[24]After all, it isn't *a priori* obvious that "stitching together" the $X(A)$s starting from any asset level forms an equilibrium path. When $X(A) = A$, it does.



(A) Constructing the interval $[S_1, S_2]$ (B) Double scaling

FIGURE 6. THRESHOLD FOR STRONG SELF-CONTROL.

Figure 6. Let S^* be the supremum asset level on $[B, A^*]$ for which $X(A) > A$; then as in the exposition for part (i), $X(S^*) = S^*$ and S^* is indeed sustainable.

By Observation 2, the function $X(A)$ on $[B, S^*]$ can be scaled and replicated as an equilibrium choice over $[S^*, S_1]$, where S_1 bears the same ratio to S^* as S^* does to B.[25] Figure 6 shows this as the dotted line with domain $[S^*, S_1]$. Because there is a poverty trap near B, the line lies below the 45^0 line to the right of B and to the right of S^*.

That said, there is one feature near S^* that cannot be replicated near B. Just to the right of S^*, it is possible to implement even smaller continuation assets by dipping into the zone to the left of S^*, and then accumulating upwards along $X(A)$ back towards S^*. Because these choices — shown by the solid line to the right of S^* in Figure 6 — favor current consumption over the future, they generate even lower equilibrium *values*, but they earn high enough *payoffs* so that they can be successfully implemented as equilibria. These lower values infiltrate up the region to the right, and do a better job of forestalling deviations at even higher asset levels. In particular, for asset levels close to S_1, the incentive constraints are relaxed and larger values of continuation assets (the solid line again) are sustainable. In particular, while S_1 is a sustainable asset level, it *also* permits accumulation: $X(S_1) > S_1$.

[25]The actual proof is considerably more complex at this point. Section 9 makes the argument formally and adds an intuitive description. Briefly, we may need to rescale S^* several times before we arrive at the equivalent of what we call S_1 here, which is shown in this exposition as a single rescaling of S^*.

This argument creates a zone (possibly a small interval, but an interval nonetheless) just above S_1, call it (S_1, S_2), over which (a) $X(A) > A$, and (b) each S_1 and S_2 is sustainable. Part (a) follows from the fact that $X(S_1) > S_1$ and that X is nondecreasing. Part (b) follows from the fact that assets just to the right of S_1 were at least "almost sustainable" by virtue of the scaling argument of Observation 2, and now must be deemed fully sustainable by virtue of the additional punishment properties that we've established in the region of S^*.

Panel B of Figure 6 now focusses fully on this zone and its implications. The following variation on Observation 2, stated and proved formally as Lemma 15 in Section 9, forms our central argument:

OBSERVATION **3.** *Suppose that S_1 and S_2 are both sustainable, and that $X(A) > A$ for all $A \in (S_1, S_2)$. Then there exists $\hat{A}$ such that $X(A) > A$ for all $A > \hat{A}$.*

The proof of the lemma is illustrated in the second panel of Figure 6. Define $\mu_i = \frac{S_i}{B}$ for $i = 1, 2$. Then for all positive integers k larger than some threshold K, the intervals $(\mu_1^k S_1, \mu_2^k S_2)$ and $(\mu_1^{k+1} S_1, \mu_2^{k+1} S_2)$ *must overlap*. It is easy to see why: $\mu_2^k S_2$ is just $\mu_2^{k+1} B$ while $\mu_1^{k+1} S_1$ is $\mu_1^{k+2} B$, and for large k it must be that μ_2^{k+1} exceeds μ_1^{k+1}.

Once this is settled, we can generate any asset level $A > \mu_1^K S_1$ by simply choosing an integer $k \geq K$, an integer m between 0 and k, and $A' \in (S_1, S_2)$ so that

$$A = \mu_1^m \mu_2^{k-m} A'.$$

But $X(A') > A'$, so that repeated application of Observation 2 proves that $X(A) > A$. That proves Observation 3.

But now the proof of the theorem is complete: by part (ii) of Proposition 2, if $X(A) > A$ for all A sufficiently large, the required threshold A_2 must exist.

6. SOME IMPLICATIONS OF THE THEORY

The main connection we emphasize in this paper is that there is a systematic link between credit limits and the ability to exercise self-control. The very same individual (psychologically speaking) can behave quite differently when he is close to the minimum level to which his assets can go. He has little power to exercise control over himself using internal rules, because the narrow margins below which he is constrained not to fall do not leave any room for suitable “punishments”. In contrast, when that individual has sufficient wealth, he can use internal rules to carry out sustained accumulation, provided, of course, that his proclivity for current consumption is not extremely high.

It is evident that while our model is not scale-neutral, there is neutrality in a modified sense. One such sense (another is given in Lemma 15) is that the *ratio* of initial asset A to the lower bound B — fully determines an individual’s ability to exercise self-control. We can rephrase all our observations in terms of this ratio. In particular, Proposition 4 can be interpreted as saying that there are two ratios μ_1 and μ_2, with $1 < \mu_1 \leq \mu_2 < \infty$, such that a poverty trap exists whenever $A/B < \mu_1$, while unlimited accumulation is possible whenever $A/B > \mu_2$.

6.1. **Ambiguous Effects of Changing Credit Limits.** A first implication of Proposition 4 is that an improvement in the credit limit has ambiguous effects, depending on initial assets A. Such an improvement lowers B. If that tips A/B over the threshold μ_2, sustained accumulation becomes possible where none was possible before. On the other hand, if A/B remains below μ_1 after the improvement, the individual will slide into an even deeper poverty trap.

6.2. **Asset-Specific Marginal Propensities to Consume.** A second implication is that the model naturally generates different marginal propensities to consume from income flows and assets. This phenomenon is studied in Hatsopoulos, Krugman and Poterba (1989), Thaler (1990) and Laibson (1997), though admittedly the empirical evidence for it may be somewhat debatable.[26] To see this, recall Section 2.1 and our interpretation there of the lower bound as some function of permanent income, presumably one that is related to the fraction of future labor income that can be seized in the event of a default. That is, if F_t stands for *financial* assets at date t and y for income at every date, then A_t is the present value of financial and labor assets:

$$A_t = F_t + \frac{\alpha y}{\alpha - 1},$$

while

$$B = \frac{\sigma \alpha y}{\alpha - 1}$$

[26]Some of the effects may be driven by variations in the different stochastic processes governing various sources of income.

for some $\lambda \in (0, 1)$. With this in mind, consider an increase in current financial assets F. Then B is unchanged, so that A/B must rise. Our proposition suggests that this will enhance self-control, so that accumulation is possible in a situation where previously it wasn't. In that case, the marginal propensity to consume out of an unforeseen change in financial assets could be "low".

In contrast, consider an equivalent jump in y, so that A rises by the same amount. Under our specification, B/y is constant so that A/B must *fall*. By the ratio interpretation of Proposition 4, self-control is damaged: the marginal propensity to consume from an unforeseen change in permanent income will be high. Indeed, even if B is a more complex increasing function of permanent income, it is only in the extreme case that B is entirely unchanged when y increases, and it is in that case that the propensities to consume out of the two asset classes will be the same. Otherwise, they will be different.

6.3. **External Versus Internal Commitments.** Our model is one that fully emphasizes internal rules for achieving self-control. An important extension is one to the case in which both internal and external commitments are available. The latter would include bank deposit schemes in which there are constraints on withdrawal, or legal commitments to make ongoing deposits (or both). Certainly, external commitments help when internal commitment fails, and this suggests that the asset-poor would have a higher demand for such arrangements.

Notice that external commitments effectively raise the value of the credit limit B, because they represent assets that cannot be drawn down. That suggests that unless *all* savings are carried out in the form of locked commitment schemes, such arrangements might damage other forms of "internal savings" as external assets accumulate. ("I have a retirement account, I don't need to save.")

That suggests that a judicious policy that takes advantage of external commitments below some asset threshold, coupled with a reliance on internal commitments when above that threshold, might be optimal. For instance, one might permit the agent to choose a particular savings targets, upon the attainment of which the lock-in is removed. To make this argument precise, we need some uncertainty in preferences or in the external environment which renders *exclusive* external arrangements infeasible.

6.4. **Who Wants External Commitments?** Finally, we comment on the economic characteristics of those individuals who might value external commitments. Clearly, these are the individuals who are asset-poor relative to their credit limit. The asset-rich would rather save on their own. The same observations would also be true of the income-poor and the income-rich provided B is unchanged across the two categories. On the other hand, these observations are reversed if B is a constant fraction of permanent income. In that case, and *controlling* for financial assets, it is the income-rich who would exhibit a greater desire for external commitment.

To be sure, the income-rich may also be asset-rich, so that the net effect is ambiguous. Nevertheless, the theory informs an empirical specification which can, in principle, be tested.

7. Numerical Analysis

8. Conclusions and Directions for Future Research

9. PROOFS

LEMMA **1.** *For any equilibrium continuation $\{x, V\}$ at A,*

$$\begin{aligned} V &\geq \left[u\left(A - \frac{B}{\alpha}\right) + \delta L(B)\right] + \frac{1-\beta}{\alpha\beta} u'\left(A - \frac{B}{\alpha}\right)(x - B) \\ &\geq u\left(A - \frac{B}{\alpha}\right) + \frac{\delta}{1-\delta} u\left(\frac{\alpha - 1}{\alpha} B\right). \end{aligned} \tag{15}$$

Proof. By (5) and the restriction that $A_t \geq B$ for any feasible asset choice at date t, we have for any feasible consumption c_t at date t,

$$u(c_t) \geq u(\nu B),$$

so that $L(A) \geq (1-\delta)^{-1} u(\nu B) > -\infty$.

Let P be the payoff associated with $\{x, V\}$. Then $P = (1-\beta) u\left(A - \frac{x}{\alpha}\right) + \beta V$ and

$$(1-\beta) u\left(A - \frac{x}{\alpha}\right) + \beta V \geq u\left(A - \frac{B}{\alpha}\right) + \beta\delta L(B),$$

because $\{x, V\}$ is an equilibrium. Noting that $u\left(A - \frac{x}{\alpha}\right) \leq u\left(A - \frac{B}{\alpha}\right)$,

$$\begin{aligned} V &\geq \left[u\left(A - \frac{B}{\alpha}\right) + \delta L(B)\right] + \frac{1-\beta}{\beta}\left[u\left(A - \frac{B}{\alpha}\right) - u\left(A - \frac{x}{\alpha}\right)\right] \\ &\geq \left[u\left(A - \frac{B}{\alpha}\right) + \delta L(B)\right] + \frac{1-\beta}{\alpha\beta} u'\left(A - \frac{B}{\alpha}\right)(x - B). \end{aligned} \tag{16}$$

By applying (16) to $A = B$ and the value $L(B)$, or (if needed) a sequence of equilibrium values in $\mathcal{V}(B)$ that converge down to $L(B)$,

$$L(B) \geq u\left(B - \frac{B}{\alpha}\right) + \delta L(B). \tag{17}$$

Combining (16) and (17), the proof is complete. ∎

Proof of Observation 1. This is an obvious consequence of Lemma 1. ∎

Proof of Proposition 1. We claim that if $\mathcal{W}$ is nonempty, has closed graph, and satisfies (8), then it generates a correspondence with the same properties. Let $\mathcal{W}'$ be the correspondence generated by $\mathcal{W}$. We first prove that $\mathcal{W}'$ is nonempty-valued. Consider the function $H_{\mathcal{W}}$ on $[B, \infty)$ defined by

$$H_{\mathcal{W}}(A) \equiv \max \mathcal{W}(A)$$

for all $A \geq B$. It is easy to see that $H_{\mathcal{W}}$ is usc. It follows that the problem

$$\max_{x \in [0, A/\alpha]} u\left(A - \frac{x}{\alpha}\right) + \beta\delta H_{\mathcal{W}}(x)$$

is well-defined and admits a solution $x(A)$ for every $A \geq B$. Define

$$W \equiv u\left(A - \frac{x(A)}{\alpha}\right) + \delta H_{\mathcal{W}}(x(A)).$$

Now observe that W satisfies (9) — pick $x = x(A)$ and $V = H_{\mathcal{W}}(x(A))$. It also satisfies (10) — for each feasible alternative x', take V' to be any element of $\mathcal{W}(x')$.

We next prove that $\mathcal{W}'$ has closed graph. Take any $\{A_n, W_n\}$ such that (i) $W_n \in \mathcal{W}'(A_n)$ for all n, and (ii) $(A_n, W_n) \to (A, W)$ as $n \to \infty$. Pick x_n feasible for A_n and value $V_n \in \mathcal{W}(x_n)$ such that (9) is satisfied. Then for any limit point (x, V) of (x_n, V_n), we have $V \in \mathcal{W}(x)$, x feasible for A, and (9) also satisfied.

To verify (10) for (A, W), pick x' feasible for A. Choose x'_n feasible for A_n with $x'_n \to x'$. $\{A_n, W_n\}$ is an equilibrium continuation, so there is $V'_n \in \mathcal{W}(x'_n)$ satisfying (10). Let V' be any limit point of $\{V'_n\}$, then $V' \in \mathcal{W}(x')$, and so (10) holds for (A, W) at x'.

With this claim in hand, consider the iterated sequence $\{\mathcal{V}_k\}$. Because $\mathcal{V}_0$ is nonempty, has closed graph, and satisfies (8), so do all the $\mathcal{V}_k$'s. Moreover, for each $t \geq 0$, it is obvious that for all $A \geq B$,

$$\mathcal{V}_k(A) \supseteq \mathcal{V}_{k+1}(A)$$

Take infinite intersections of these nested compact sets (at each A) to argue that

$$\mathcal{V}(A) \equiv \bigcap_{t=0}^{\infty} \mathcal{V}_k(A)$$

is nonempty and compact-valued for every A. Indeed, $\mathcal{V}$ has compact graph on any compact interval $[B, D]$,[27] and therefore it has closed graph everywhere. By picking $V \in \mathcal{V}(A)$ and taking limits of continuation assets, values and punishments as $k \to \infty$, it is immediate that $\mathcal{V}$ generates itself and contains all other correspondences that do, so it is our equilibrium correspondence. ■

[27]On any compact interval, the (restricted) graphs of the $\mathcal{V}_k$'s are compact and their infinite intersection is the graph of $\mathcal{V}$ on the same interval, which must then be compact.

In the light of Proposition 1, define $H(A)$ and $L(A)$ to be the maximum and minimum values of the equilibrium value correspondence $\mathcal{V}$. Because the graph of $\mathcal{V}$ is closed, H is usc and L is lsc. "Fill up" $L(A)$ into a correspondence by defining

$$\mathcal{L}(A) \equiv \{L | L \text{ is a limit of } L(A^n) \text{ for some sequence } A^n \to A\}.$$

It is obvious that $\mathcal{L}$ has closed graph and coincides with L at all points of continuity of the latter. Define $\hat{L}(A) \equiv \max \mathcal{L}(A)$, and the "best deviation payoff" at A by

$$D(A) = \max_{y \in [B, \alpha(1-v)A]} u\left(A - \frac{y}{\alpha}\right) + \beta\delta\hat{L}(y). \tag{18}$$

Let $d(A)$ denote a generic asset choice solving (18), and denote by $d^*(A)$ the largest of these. All these objects are well-defined.

Obviously, $D(A)$ is increasing. D does not necessarily use worst punishments everywhere, but nonetheless a deviant can get payoff arbitrarily close to $D(A)$. The interpretation of D as "best deviation" is formalized in Lemma 2 below.

Each feasible asset sequence $\{A_t\}$ (or path) generates a corresponding sequence of values $\{V_t\}$: $V_t \equiv \sum_{s=t}^{\infty} \delta^{s-t} u\left(A_t - \frac{A_{t+1}}{\alpha}\right)$ for every $t \geq 0$.

LEMMA **2.** *A path $\{A_t\}$ is an equilibrium if and only if*

$$u\left(A_t - \frac{A_{t+1}}{\alpha}\right) + \beta\delta V_{t+1} \geq D(A_t) \tag{19}$$

for all t, where $\{V_t\}$ is the sequence of values generated by the path.

Proof. Sufficiency is a consequence of the one-shot deviation principle, given that D is obviously the supremum of all payoffs that can be realized following a single deviation and then an application of the worst equilibrium continuation value. To prove necessity, note that if $\{A_t\}$ is an equilibrium path, then for every t,

$$u\left(A_t - \frac{A_{t+1}}{\alpha}\right) + \beta\delta V_{t+1} \geq \sup_{y \in [B, \alpha(1-v)A_t] \setminus A_{t+1}, L \in \mathcal{L}(y)} u\left(A_t - \frac{y}{\alpha}\right) + \beta\delta L$$

But $V_{t+1} \geq L(A_{t+1})$ for each t and $\hat{L}(y) = \max \mathcal{L}(y)$, so

$$u\left(A_t - \frac{A_{t+1}}{\alpha}\right) + \beta\delta V_{t+1} \geq \sup_{y \in [B, \alpha(1-v)A_t]} u\left(A_t - \frac{y}{\alpha}\right) + \beta\delta\hat{L}(y) = D(A_t).$$

■

LEMMA **3.** *If $d(A)$ solves (18), it is an equilibrium asset choice at A.*

Proof. Specify continuations at A by $\{d(A), \hat{L}(A)\}$ (noting that $\hat{L}(A)$ id an equilibrium value at A, because $\mathcal{V}$ is uhc) and by the continuation values $L(x)$ if any other asset level is chosen, and apply (19) in Lemma 2. ■

Moreover, standard arguments prove

LEMMA **4.** *Let $d(A)$ be a generic asset choice that solves* (18). *If $A_1 < A_2$, then $d(A_1) \leq d(A_2)$. Moreover, a maximal solution $d^*(A)$ is well-defined, and this choice is nondecreasing and usc in A.*

LEMMA **5.** *$L(A)$ is increasing on $[B, \infty)$.*

Proof. Let $A'' > A' \geq B$. Consider the equilibrium that generates value $L(A'')$ starting from A'', with associated continuation $\{A_1'', V''\}$. By Lemma 2,

(20) $$u\left(A'' - \frac{A_1''}{\alpha}\right) + \beta\delta V'' \geq u\left(A'' - \frac{x}{\alpha}\right) + \beta\delta\hat{L}(x)$$

for $x \in [B, \alpha(1-\nu)A'']$. (20) implies in particular that $V'' > \hat{L}(x)$ for $x < A_1''$, so

(21) $$L(A'') = u\left(A'' - \frac{A_1''}{\alpha}\right) + \delta V'' > u\left(A'' - \frac{x}{\alpha}\right) + \delta\hat{L}(x)$$

for all $x < A_1''$. Now construct an equilibrium from A', as follows. The choice A_1'' (if feasible) is followed by the value V'', while every other $x \in [B, \alpha(1-\nu)A']$ is followed by $\hat{L}(x)$.[28] Because u is strictly concave and $A' < A''$, it follows from (20) that

(22) $$u\left(A' - \frac{A_1''}{\alpha}\right) + \beta\delta V'' > u\left(A' - \frac{x}{\alpha}\right) + \beta\delta\hat{L}(x)$$

for $x \in (A_1'', \alpha(1-\nu)A']$ (assuming this set is non-empty). Choose continuation $\{y, V\}$ at A' to maximize payoff over these specifications. That continuation must be an equilibrium, and by (22), $y \leq A_1''$. If $y < A_1''$, then (21) implies

$$L(A'') > u\left(A'' - \frac{y}{\alpha}\right) + \delta\hat{L}(A_1') > u\left(A' - \frac{y}{\alpha}\right) + \delta\hat{L}(y) \geq L(A'),$$

and if $y = A_1''$, then again

$$L(A'') = u\left(A'' - \frac{A_1''}{\alpha}\right) + V'' > u\left(A' - \frac{y}{\alpha}\right) + V'' \geq L(A').$$

So in both cases, $L(A'') > L(A')$, as desired. ■

[28]Note that $\hat{L}(x)$ is indeed an equilibrium value at x because $\mathcal{V}$ has closed graph.

Lemma 5 makes it easy to visualize the incentive constraint embodied in $D(A)$. At any x, $\hat{L}(x)$ is just the limit of $L(x')$ as $x' \downarrow x$. The following lemma is an immediate consequence of the analysis so far:

LEMMA **6.** $\hat{L}(A)$ *is increasing and usc (and therefore right-continuous).*

It will be convenient to define the *maintenance value* of an asset level A by

$$V^s(A) \equiv \frac{1}{1-\delta} u\left(\frac{\alpha-1}{\alpha}A\right),$$

and the *maintenance payoff* by

$$P^s(A) \equiv \left[1 + \frac{\beta\delta}{1-\delta}\right] u\left(\frac{\alpha-1}{\alpha}A\right).$$

Say that an asset level S is *sustainable* if there is a stationary equilibrium path from S, or equivalently (by Lemma 2) if $P^s(A) \geq D(A)$.

LEMMA **7** (Observation 2 in main text)**.** (a) *Let* $S > B$ *be sustainable. Define* $\mu = S/B$. *Then if* $\{A_t^*\}$ *is an equilibrium path from* A_0*, so is* $\{\mu A_t^*\}$ *from* μA_0.

(b) *For all t with* $\mu A_t^* > B$ *and for every* $A < S$,

$$u\left(\mu A_t^* - \frac{\mu A_{t+1}^*}{\alpha}\right) + \beta \sum_{s=t+1}^{\infty} \delta^{s-t} u\left(\mu A_s^* - \frac{\mu A_{s+1}^*}{\alpha}\right) > u\left(\mu A_t^* - \frac{A}{\alpha}\right) + \beta\delta\hat{L}(A).$$

Proof. Part (a). Let policy ϕ sustain $\{A_t^*\}$ from A_0. Define a new policy ψ:

(i) For any $h_t = (A_0 \ldots A_t)$ with $A_s \geq S$ for $s = 0, \ldots, t$, let $\psi(h_t) = \mu\phi\left(\frac{h_t}{\mu}\right)$.

(ii) For h_t with $A_s < S$ for some smallest $k \leq t$, define $h'_{t-k} = (A_k \ldots A_t)$. Let $\psi(h_t) = \phi_\ell(h'_{t-k})$, where ϕ_ℓ is the equilibrium policy with value $L(A_k)$ at A_k.

For any history h_t with $A_s \geq S$ for $s = 1, \ldots, t$, the asset sequence generated through subsequent application of ψ is the same as the sequence generated through repeated application of ϕ from $\frac{h_t}{\mu}$, but scaled up by the factor μ. It follows that

(23) $$P_\psi(h_t) = \mu^{1-\sigma} P_\phi\left(\frac{h_t}{\mu}\right) \text{ and } V_\psi(h_t) = \mu^{1-\sigma} V_\phi\left(\frac{h_t}{\mu}\right).$$

We now show that ψ is an equilibrium.

First, consider any h_t such that $A_k < S$ at some first $k \leq t$. Then as of period k the policy function ψ shifts to the equilibrium with value $L(A_k)$. So $\psi(h_t)$ is optimal given the continuation policy function.

Next consider any h_t such that $A_s \geq S$ for all $s \leq t$. Consider, first, any deviation to $A \geq S$. Note that h_t/μ is a feasible history under the equilibrium ϕ, while the deviation to $(A/\mu) \geq (S/\mu) = B$ is also feasible. It follows that

$$P_\phi\left(\frac{h_t}{\mu}\right) \geq u\left(\frac{A_t}{\mu} - \frac{A}{\mu\alpha}\right) + \beta\delta V_\phi\left(\frac{h_t}{\mu}\right).$$

Multiplying through by $\mu^{1-\sigma}$ and using (23), we see that

$$P_\psi(h_t) \geq u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta V_\psi(h_t.A), \tag{24}$$

which shows that no deviation to $A \geq S$ can be profitable.

Now consider a deviation to $A < S$. Because S is sustainable,

$$P^s(S) \geq u\left(S - \frac{A}{\alpha}\right) + \beta\delta\hat{L}(A). \tag{25}$$

At the same time, (24) applied to $A = S$ implies

$$P_\psi(h_t) \geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta V_\psi(h_t.S). \tag{26}$$

Using (23) along with $L(B) \geq V^s(B)$ (see Observation 1), (26) becomes

$$\begin{aligned}
P_\psi(h_t) &\geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta\mu^{1-\sigma}V_\phi\left(\frac{h_t}{\mu}.B\right) \\
&\geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta\mu^{1-\sigma}L(B) \\
&\geq u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta\mu^{1-\sigma}V^s(B) \\
&= u\left(A_t - \frac{S}{\alpha}\right) + \beta\delta V^s(S) \\
&= \left[u\left(A_t - \frac{S}{\alpha}\right) - u\left(S\left(1 - \frac{1}{\alpha}\right)\right)\right] + P^s(S). \qquad (27)
\end{aligned}$$

Combining (25) and (27) and using the concavity of u (along with $S > A$),

$$\begin{aligned} P_\psi(h_t) &\geq \left[u\left(A_t - \frac{S}{\alpha}\right) - u\left(S\left(1-\frac{1}{\alpha}\right)\right)\right] + u\left(S - \frac{A}{\alpha}\right) + \beta\delta\hat{L}(A) \\ &= \left[u\left(A_t - \frac{S}{\alpha}\right) - u\left(S - \frac{S}{\alpha}\right)\right] - \left[u\left(A_t - \frac{A}{\alpha}\right) - u\left(S - \frac{A}{\alpha}\right)\right] \\ &\quad + u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta\hat{L}(A) \\ &\geq u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta\hat{L}(A) \\ &= u\left(A_t - \frac{A}{\alpha}\right) + \beta\delta V_\psi(h_t.A), \end{aligned} \tag{28}$$

where the second inequality follows from the concavity of u and the fact that $A < S$. It follows that the deviation A is unprofitable, so that ψ is an equilibrium.

Part (b). The second inequality in (28) holds strictly when $A_t > S$ and $A < S$, because u is strictly concave. Apply this observation to the on-path history in which $A_t = \mu A_t^* > S$ by assumption. ∎

LEMMA **8.** *For any asset level A and any path $\{A_t\}$ with $A_t \leq A$ for all $t \geq 0$,*

$$V^s(A) - \sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right) \geq u'\left(\frac{\alpha-1}{\alpha}A\right)\left(\delta - \frac{1}{\alpha}\right)(A - A_1) \geq 0. \tag{29}$$

Proof. Let Δ stand for the left hand side of (29); then

$$\begin{aligned} \Delta &= \sum_{t=0}^{\infty} \delta^t \left[u\left(\frac{\alpha-1}{\alpha}A\right) - u\left(A_t - \frac{A_{t+1}}{\alpha}\right)\right] \\ &\geq u'\left(\frac{\alpha-1}{\alpha}A\right) \sum_{t=0}^{\infty} \delta^t \left[A - \frac{A}{\alpha} - A_t + \frac{A_{t+1}}{\alpha}\right] \\ &= u'\left(\frac{\alpha-1}{\alpha}A\right) \sum_{t=0}^{\infty} \delta^t \left[(A - A_t) - \frac{A - A_{t+1}}{\alpha}\right] \\ &= u'\left(\frac{\alpha-1}{\alpha}A\right) \left[(A - A_0) + \left(\delta - \frac{1}{\alpha}\right) \sum_{t=0}^{\infty} \delta^t (A - A_{t+1})\right] \\ &\geq u'\left(\frac{\alpha-1}{\alpha}A\right)\left(\delta - \frac{1}{\alpha}\right)(A - A_1) \geq 0, \end{aligned}$$

where the first inequality uses the concavity of u and the last two use $\delta\alpha > 1$. ■

Define $X(A)$ and $Y(A)$ to be the largest and smallest equilibrium asset choices respectively at A.

LEMMA **9.** *$X(A)$ and $Y(A)$ are well-defined and non-decreasing, and X is usc.*

Proof. It is easy to see that $X(A)$ (resp. $Y(A)$) is the largest (resp. smallest) value of $A' \in [B, \alpha(1-\upsilon)A]$ satisfying

$$u\left(A - \frac{A'}{\alpha}\right) + \beta\delta H(A') \geq D(A) \geq u\left(A - \frac{y}{\alpha}\right) + \beta\delta\hat{L}(y) \tag{30}$$

for all $y \in [B, \alpha(1-\upsilon)A]$. $X(A)$ and $Y(A)$ are well-defined because H is usc. To show that X is non-decreasing, pick $A_1 < A_2$. If $u\left(A_2 - X(A_1)/\alpha\right) + \beta\delta H(X(A_1)) \geq D(A_2)$, then we are done. Otherwise there is $x' \in [B, \alpha(1-\upsilon)A_2]$ such that

$$u\left(A_2 - \frac{X(A_1)}{\alpha}\right) + \beta\delta H(X(A_1)) < u\left(A_2 - \frac{x'}{\alpha}\right) + \beta\delta\hat{L}(x'), \tag{31}$$

which implies

$$u\left(A_2 - \frac{x'}{\alpha}\right) - u\left(A_2 - \frac{X(A_1)}{\alpha}\right) > \beta\delta\left[H(X(A_1)) - \hat{L}(x')\right] \tag{32}$$

There are two cases to consider: (i) $x' \leq X(A_1)$, and (ii) $x' > X(A_1)$. In case (i), x' is feasible under A_1, so that

$$u\left(A_1 - \frac{x'}{\alpha}\right) - u\left(A_1 - \frac{X(A_1)}{\alpha}\right) \leq \beta\delta\left[H(X(A_1)) - \hat{L}(x')\right] \tag{33}$$

But (32) and (33) together contradict the concavity of u.

In case (ii), we combine (30) and (31) to see that

$$\begin{aligned} u\left(A_2 - \frac{x'}{\alpha}\right) + \beta\delta\hat{L}(x') &> u\left(A_2 - \frac{X(A_1)}{\alpha}\right) + \beta\delta H(X(A_1)) \\ &\geq u(A_2 - y) + \beta\delta\hat{L}(y) \end{aligned} \tag{34}$$

for all $y \leq X(A_1)$. We now construct an equilibrium starting from A_2 as follows: any choice $A < X(A_1)$ is followed by the continuation equilibrium generating $L(A)$, and any choice $A \geq X(A_1)$ is followed by the continuation equilibrium generating $H(A)$. Because H is usc, there exists some z^* that maximizes $u\left(A_2 - \frac{z}{\alpha}\right) + \beta\delta H(z)$ on $[X(A_1), \alpha(1-\upsilon)A_2]$; in light of (34) and the fact that $u\left(A_2 - \frac{x}{\alpha}\right) + \beta\delta H(x) \geq$

$u\left(A_2 - \frac{x}{\alpha}\right) + \beta\delta L(x)$, all choices $A < X(A_1)$ are strictly inferior to z^*. Thus z^* is an equilibrium choice at A_2, so that $X(A_2) \geq z^* \geq X(A_1)$.

To show that $Y(A)$ is non-decreasing, pick $A_1 < A_2$. If $Y(A_2) \geq \alpha[1-\upsilon]A_1$, we're done, so suppose that $Y(A_2) < \alpha[1-\upsilon]A_1$. Construct an equilibrium from A_1 as follows: assign the continuation $\{A', H(A')\}$, where A' solves

$$\max_{A \in [B, Y(A_2)]} u\left(A_1 - \frac{A}{\alpha}\right) + \beta\delta H(A)$$

(Because H is usc, a solution exists.) For any other $A \in (Y(A_2), \alpha[1-\upsilon]A_1]$, assign the value $L(A)$; for $A \in [B, Y(A_2)]$, assign $H(A)$. We claim that A' maximizes payoff over all these specifications. It certainly does so over asset choices in $[B, Y(A_2)]$, by construction. For $A > Y(A_2)$,

$$u\left(A_2 - \frac{Y(A_2)}{\alpha}\right) + \beta\delta H(Y(A_2)) \geq u\left(A_2 - \frac{A}{\alpha}\right) + \beta\delta \hat{L}(A),$$

so by the concavity of u,

$$u\left(A_1 - \frac{Y(A_2)}{\alpha}\right) + \beta\delta H(Y(A_2)) > u\left(A_1 - \frac{A}{\alpha}\right) + \beta\delta \hat{L}(A),$$

which proves the claim. Because $A' \leq Y(A_2)$, it follows that $Y(A_1) \leq Y(A_2)$.

Finally, we show that X is usc. For any $A^* \geq B$, $\lim_{A \uparrow A^*} X(A) \leq X(A^*)$ because $X(A)$ is nondecreasing. Now consider any decreasing sequence $A^k \downarrow A^*$, and let X^* denote the (well-defined) limit of $X(A^k)$. For each k, $u\left(A^k - X(A^k)/\alpha\right) + \beta\delta H(X(A^k)) \geq D(A^k)$. Because H is usc and $D(A)$ is nondecreasing, $u\left(A^* - X^*/a\right) + \beta\delta H(X^*) \geq \lim_{k\to\infty} D(A^k) \geq D(A^*)$. That implies $X(A^*) \geq X^* = \lim_{A \downarrow A^*} X(A)$. (In fact, because $X(A)$ is non-decreasing, $X(A^*) = \lim_{A \downarrow A^*} X(A)$.) ■

LEMMA **10.** *If $X(A) = A$, then A is sustainable.*

Proof. Let $\{A_t\}$ be an equilibrium path from A with $A_1 = A$. Then

$$u\left(\frac{\alpha - 1}{\alpha} A\right) + \beta\delta V_1 \geq D(A).$$

by Lemmas 2 and 9. By Lemma 8, $V_1 \leq (1-\delta)^{-1} u\left(\frac{\alpha-1}{\alpha}A\right)$. Using this in the inequality above, we see that $P^m(A) \geq D(A)$, so that A is sustainable. ■

LEMMA **11.** *In the nonuniform case, $\beta\delta(\alpha - 1)/(1-\delta) < 1$.*

Proof. We claim that if $\beta\delta(\alpha-1)/(1-\delta)\geq 1$, then there exists a linear Markov equilibrium policy function $\phi(A)=kA$ with $k>1$, which implies uniformity.

To this end, assume that all "future selves" employ the policy function $\phi(A)=kA$ with $k\in[1,\alpha]$ for all $A\geq B$. The individual's current problem is to solve

$$\max_{x\in[B,\alpha(1-\upsilon)A]}\frac{1}{1-\sigma}\left[\left(A-\frac{x}{\alpha}\right)^{1-\sigma}+\beta\delta Qx^{1-\sigma}\right]$$

where

$$Q\equiv\frac{(\alpha-k)^{1-\sigma}}{\alpha^{1-\sigma}\left(1-\delta k^{1-\sigma}\right)}\tag{35}$$

The corresponding necessary and sufficient first-order condition is

$$\frac{1}{\alpha}\left(A-\frac{x}{\alpha}\right)^{-\sigma}=\beta\delta Qx^{-\sigma}.$$

After some manipulation, we obtain

$$\frac{A}{x}=\frac{1}{\alpha}+\left(\frac{1}{\alpha\beta\delta Q}\right)^{1/\sigma}\equiv\frac{1}{k^*}\tag{36}$$

Note that $x=k^*A$. Accordingly, the policy function is an equilibrium if $k^*=k$. Substituting (35) into (36) and rearranging yields

$$k^{\sigma}=\alpha\beta\delta+(1-\beta)\,\delta k\tag{37}$$

Define $\Lambda(k)\equiv k^{1-\sigma}$ and $\Phi(k)=\alpha\beta\delta+(1-\beta)\,\delta k$. Notice that $\Lambda(1)\leq\Phi(1)$ (given that $\beta\delta(\alpha-1)/(1-\delta)\geq 1$), and $\Lambda(\alpha)>\Phi(\alpha)$ (given the transversality condition $\delta\alpha^{1-\sigma}<1$). By continuity, it follows that there exists a solution on the interval $[1,\alpha)$, which establishes the claim and hence the lemma. ■

LEMMA **12.** *Under nonuniformity, the problem*

$$\max_{x\in[0,\alpha(1-\upsilon)A]}\left[u\left(A-\frac{x}{\alpha}\right)+\beta\delta V^s(x)\right].$$

has a unique, continuous solution $x(A)$ with $x(A)=\Gamma A$, where $0<\Gamma<1$. Moreover, the maximand is strictly decreasing in x for all $x\geq x(A)$.

Proof. It is obvious that the maximand is a continuous, strictly concave function, Therefore it has a unique, continuous solution $x(A)$ for each A. Moreover, by strict concavity, the maximand must strictly decline in x for all $x\geq x(A)$.

Define $\xi = \beta\delta(\alpha-1)/(1-\delta)$. By nonuniformity and Lemma 11, we know that $\xi < 1$. Routine computation reveals that $x(A) = \Gamma A$, where

$$\Gamma = \frac{\alpha}{1 + \xi^{-\frac{1}{\sigma}}(\alpha - 1)}$$

which (given $\sigma > 0$ and $\xi < 1$) implies $\Gamma < 1$. ■

LEMMA **13.** *For any $A_0 \geq B$, maximize $\sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right)$, subject to $A_{t+1} \in [B, \alpha(1-\upsilon)A_t]$, and $A_{t+1} \leq X(A_t)$ for all $t \geq 0$. Then a solution exists, and any solution path $\{A_t^*\}$ is also an equilibrium path starting from A_0.*

Proof. u is continuous and $X(A_t)$ is usc (Lemma 9), so a solution $\{A_t^*\}$ (with associated values $\{V_t^*\}$) exists. Consider an equilibrium path from date t, call it $\{A_\tau\}$, sustaining $X(A_t^*)$ at A_t^* and providing continuation value $H(X(A_t^*))$ thereafter. This path necessarily satisfies $A_{\tau+1} \leq X(A_\tau)$ for all $\tau \geq t$, so the definition of $\{A_t^*\}$ implies that

$$u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) + \delta V_{t+1}^* \geq u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) + \delta H(X(A_t^*)) \tag{38}$$

Also, because $A_{t+1}^* \leq X(A_t^*)$ and $\beta < 1$, we have

$$\left(\frac{1}{\beta} - 1\right) u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) \geq \left(\frac{1}{\beta} - 1\right) u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) \tag{39}$$

Adding (38) to (39) and multiplying through by β, we obtain

$$u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) + \beta\delta V_{t+1}^* \geq u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) + \beta\delta H(X(A_t^*)) \tag{40}$$

Now, $\{X(A_t^*), H(X(A_t^*)\}$ is supportable at A_t^*, so

$$u\left(A_t^* - \frac{X(A_t^*)}{\alpha}\right) + \beta\delta H(X(A_t^*)) \geq D(A_t^*) \tag{41}$$

Combining (40) and (41), we obtain

$$u\left(A_t^* - \frac{A_{t+1}^*}{\alpha}\right) + \delta V_{t+1}^* \geq D(A_t^*)$$

for all $t \geq 0$, which shows that $\{A_t\}$ is an equilibrium path. ■

LEMMA **14.** *Suppose that for some $A^* > 0$, $X(A) > A$ for all $A \geq A^*$. Then starting from any $A \geq A^*$, there is an equilibrium path with monotonic and unbounded accumulation, so that strong self-control is possible. Moreover, some such equilibrium path maximizes value among all equilibrium paths from A.*

Proof. We first claim that for any $A > A^*$ with $\lim_{A' \uparrow A} X(A') = A$, there is $\epsilon > 0$ with

$$X(A') = A \text{ for } A' \in (A - \epsilon, A). \tag{42}$$

Suppose on the contrary that there is $A > A^*$ and $\eta > 0$ such that $A' < X(A') < A$ for all $A' \in (A - \eta, A)$. Because $X(A) > A$, Lemma 13 and $\delta\alpha > 1$ together imply

$$H(A) > V^s(A) + \gamma \tag{43}$$

for some $\gamma > 0$.[29] Consider any equilibrium continuation $\{X(A'), V_1\}$ from $A' \in (A - \eta, A)$. Because $A'' < X(A'') < A$ for all A'' in that interval, $A'_t < A$ for the resulting equilibrium path. It follows from Lemma 8 that $V^s(A) \geq V_1$. Combining this inequality with (43) and noting that $X(A') \to A$ as $A' \to A$,

$$u\left(A' - \frac{A}{\alpha}\right) + \beta\delta H(A) > u\left(A' - \frac{X(A')}{\alpha}\right) + \beta\delta V_1 \geq D(A')$$

for all $A' < A$ but close to A. So all such A' possess an equilibrium continuation of $\{A, H(A)\}$, which contradicts $X(A') < A'$, and establishes the claim.

We now complete the proof by claiming that any path $\{A_t\}$ from $A \geq A^*$ which solves the problem of Lemma 13 involves monotonic and unbounded accumulation. Suppose this assertion is false. Then at least one of the following must be true:

(i) there exists some date τ such that $A_\tau \geq A_{\tau+1} \leq A_{\tau+2}$, and/or

(ii) the sequence $\{A_t\}$ converges to some finite limit.

Let $\{c_t\}$ be the consumption sequence generated by $\{A_t\}$. In case (i), $c_\tau \geq c_{\tau+1}$. Recalling that $\delta\alpha > 1$, we therefore have

$$u'(c_\tau) < \delta\alpha u'(c_{\tau+1}). \tag{44}$$

Moreover, because $X(A_\tau) > A_\tau$ and $A_\tau \geq A_{\tau+1}$, we have

$$A_{\tau+1} < X(A_\tau). \tag{45}$$

In case (ii), there exists T such that, for $\tau > T_1$, (44) again holds because c_τ and $c_{\tau+1}$ are close. As far as (45) is concerned, there are two subcases to consider:

(a) There is $\tau > T$ with $A_{\tau+1} \leq A_\tau$. Here, (45) holds because $X(A_\tau) > A_\tau \geq A_{\tau+1}$.

[29] If $\delta\alpha > 1$ and $X(A) > A$, then the problem of Lemma 13 isn't solved by the stationary path from A.

(b) For $t > T$, A_t is strictly increasing with limit $\bar{A} < \infty$. If $\lim_{t\to\infty} X(A_t) > \bar{A}$, (45) plainly holds for some τ sufficiently large. Otherwise $\lim_{t\to\infty} X(A_t) = \bar{A}$. But in this case, we know from the first claim above that for some τ, $X(A_\tau) = \bar{A} > A_{\tau+1}$, so that (45) holds yet again for some τ sufficiently large.

In short, (44) and (45) always hold (for some τ). Now alter the path $\{A_t\}$ by increasing the period-τ asset level from $A_{\tau+1}$ to $A_{\tau+1} + \eta$, leaving asset levels unchanged for all other periods. Because $X(A)$ is non-decreasing, $A_{\tau+2} \leq X(A_{\tau+1} + \eta)$, and for small η we have $A_{\tau+1} + \eta < X(A_\tau)$ by (45); thus, the new path is feasible and also satisfies the constraints that define the value-maximizing path $\{A_t\}$. Taking the derivative of period-τ value with respect to η,

$$\frac{dV_\tau}{d\eta} = \delta^\tau \left[-u'(c_\tau)\frac{1}{\alpha} + \delta u'(c_{\tau+1})\right] > 0,$$

where the inequality holds as a consequence of (44). This contradicts the definition of $\{A_t\}$ as a path that solves the problem in Lemma 13, and so establishes the lemma. ■

Proof of Proposition 2. Part (i) is obvious. "Only if" in part (ii) is also obvious, while "if" follows from Lemma 14. Part (iii) is a consequence of the fact that X is usc, while part (iv) once again is obvious. ■

Proof of Proposition 4, part (i). First suppose that there is $\epsilon > 0$ with $X(A) \geq A$ on $[B, B+\epsilon]$. By nonuniformity, $X(A') < A'$ for some A'. X is nondecreasing and usc, so $X(S) = S$ for some $S > B$, with $X(A') < A'$ for some $A' \in (S, S+\epsilon')$, for every $\epsilon' > 0$.[30] By Lemma 10, S is sustainable. Define $\mu \equiv S/B$. By Lemma 7 (a), $\mu X(A'/\mu)$ is an equilibrium choice for every $A' \in [S, S+\mu\epsilon]$. But then $X(A') \geq \mu X(A'/\mu) \geq A'$ for all such A', a contradiction.

In particular, $X(B) = 0$, and for all $\epsilon > 0$, there exists $A_\epsilon \in (B, B+\epsilon)$ such that $X(A_\epsilon) < A_\epsilon$. But if the result is false, there is also $A'_\epsilon \in (B, B+\epsilon)$ with $X(A'_\epsilon) \geq A'_\epsilon$. Because $X(A)$ is nondecreasing, these observations imply the existence of $S_\epsilon \in (B, B+\epsilon)$ such that $X(S_\epsilon) = S_\epsilon$. By Lemma 10, S_ϵ is sustainable for all $\epsilon > 0$. But

$$D(S_\epsilon) \geq u\left(S_\epsilon - \frac{B}{\alpha}\right) + \beta\delta L(B) \geq u\left(S_\epsilon - \frac{B}{\alpha}\right) + \beta\delta V^s(B) > P^m(S_\epsilon)$$

for ϵ sufficiently small, by Lemma 12. This is a contradiction. ■

[30]Take S to be the infimum of all A with $X(A) < A$.

LEMMA **15** (Observation 3 in main text)**.** *Suppose that asset levels S_1 and S_2, with $S_1 < S_2$, are both sustainable, and that $X(A) > A$ for all $A \in (S_1, S_2)$. Then there exists $A^* \geq B$ such that $X(A) > A$ for all $A > A^*$.*

Proof. Let $\mu_i \equiv S_i/B$ for $i = 1, 2$; then $\mu_1 < \mu_2$. We claim that there is $A^* \geq B$ such that for all $A > A^*$, there are $\tilde{A} \in (S_1, S_2)$ and integers $(m, n) \geq 0$ with $A = \mu_1^n \mu_2^m \tilde{A}$.

We first show that there is A^* such that for all $A > A^*$, $A \in (\mu_1^k S_1, \mu_2^k S_2)$ for some k. Because $\mu_1 < \mu_2$, there is an integer ℓ with $\mu_1^{k+2} < \mu_2^{k+1}$ for all $k \geq \ell$. For all such k, $(\mu_1^k S_1, \mu_2^k S_2) = (\mu_1^k S_1, \mu_2^{k+1} B)$ overlaps with $(\mu_1^{k+1} S_1, \mu_2^{k+1} S_2) = (\mu_1^{k+2} B, \mu_2^{k+1} S_2)$. So $\cup_{k=\ell}^{\infty} (\mu_1^k S_1, \mu_2^k S_2) = \left(\mu_1^{\ell} S_1, \infty\right)$. Take A^* to be any number greater than $\mu_1^{\ell} S_1$.

Next we show that for each integer $k \geq 1$ and $A \in (\mu_1^k S_1, \mu_2^k S_2)$, there is $\tilde{A} \in (S_1, S_2)$ along with an integer $m \in \{0, \ldots, k\}$ such that $A = \mu_1^m \mu_2^{k-m} \tilde{A}$. Divide the interval $(\mu_1^k S_1, \mu_2^k S_2)$ (which is the same as the interval $(\mu_1^{k+1} B, \mu_2^{k+1} B)$) into a sequence of semi-open sub-intervals (preceded by an open interval) that coincide at their endpoints: $(\mu_1^{k+1} B, \mu_1^k \mu_2 B)$, $[\mu_1^k \mu_2 B, \mu_1^{k-1} \mu_2^2 B)$, $\ldots$, $[\mu_1 \mu_2^k B, \mu_2^{k+1} B)$. A must lie in one of these intervals; call it $[\mu_1^{m+1} \mu_2^{k-m} B, \mu_1^m \mu_2^{k-m+1} B)$, which we can rewrite as $[\mu_1^m \mu_2^{k-m} S_1, \mu_1^m \mu_2^{k-m} S_2)$. (The left edge is open if it is the first interval.) Thus, setting $\tilde{A} = A \mu_1^{-m} \mu_2^{m-k}$, we have $\tilde{A} \in (S_1, S_2)$ and $A = \mu_1^m \mu_2^{k-m} \tilde{A}$, as desired.

To complete the proof, pick any $A > A^*$ along with some $\tilde{A} \in (S_1, S_2)$ and $m \in \{0, \ldots, k\}$ for which $A = \mu_1^m \mu_2^{k-m} \tilde{A}$. By repeated application of Lemma 7 (a), we see that $X(A) \geq \mu_1^m \mu_2^{k-m} X(\tilde{A})$; noting that $X(\tilde{A}) > \tilde{A}$, we have $X(A) > A$. ■

Let us refer to the assertion of Proposition 4, part (ii), as the Conclusion. Lemma 15 (together with Lemma 14) implies the Conclusion. Via Lemma 15, several other situations also imply the Conclusion. Define $E(A) \equiv P^s(A) - D(A)$.

LEMMA **16.** $E(A) > 0$ *for some* $A > B$ *implies the Conclusion.*

Proof. Because u is continuous and D is increasing, there is an interval $[S_1, S_2]$ such that $E(A') > 0$ for all $A' \in [S_1, S_2]$ (e.g., take $S_2 = A$ and S_1 to be an asset level slightly below S_2). Clearly, S_1 and S_2 are both sustainable (indeed, every $A \in [S_1, S_2]$ is).

For each $A \in [S_1, S_2)$, define x_A as the largest value in $[S_1, S_2]$ satisfying

$$u\left(A - \frac{x_A}{\alpha}\right) + \beta\delta V^s(x_A). \tag{46}$$

Because $E(A) > 0$, we have $x_A > A$. Moreover, because $E(x_A) > 0$, we know that x_A is sustainable. So (46) and Lemma 2 imply the existence of an equilibrium starting from A in which assets increase to x_A immediately and then remain at x_A forever. It follows that $X(A) \geq x_A > A$ for all $A \in (S_1, S_2)$. Therefore the condition of Lemma 15 is satisfied: there are assets S_1 and S_2 with $S_1 < S_2$, both sustainable, with $X(A') > A'$ for all $A' \in (S_1, S_2)$. The Conclusion follows. ■

Say that a sustainable asset S is *isolated* if there is an interval around S with no other sustainable asset in that interval.

LEMMA **17.** *If S is sustainable and not isolated, and there exists $A^* > S$ with $X(A^*) > A^*$, then the Conclusion is true.*

Proof. There is $A^* > S$ with $X(A^*) > 0$. If $X(A') > A'$ for all $A' \geq A^*$, the Conclusion follows (Lemma 14). Otherwise, $X(A') \leq A'$ for some $A' > A^*$. Because X is nondecreasing, there is $S^* > A^*$ such that $X(S^*) = S^*$, and $X(A') > A'$ for all $A' \in [A^*, S^*)$. By Lemma 10, S^* is sustainable.

Suppose S is sustainable but not isolated. Then for every $\epsilon > 0$, there is sustainable S' with $|S' - S| < \epsilon$. Let $\mu \equiv S/B$ and $\mu' \equiv S'/B$. By Lemma 7 (a), $S_1 \equiv \mu S^*$ and $S_2 \equiv \mu' S^*$ are sustainable. Remember that $X(A') > A'$ for all $A' \in [A^*, S^*)$. Using this information, it is easy to see that if S and S' are close enough (say $S < S'$ without loss of generality), $X(A) > A$ for all $A \in (S_1, S_2)$. But now all the conditions of Lemma 15 are met, so that the Conclusion follows. ■

Part (i) of the proposition, along with Lemmas 14 and 15, allow us to piece together the following construction, *on the provisional assumption that the Conclusion is false*. $X(A)$ starts out below A near B, and then there is an infimum value — call it A_* — for all A with $X(A) > A$. There must be an interval to the right of A_* with $X(A) > A$; if not, sustainable stocks cannot all be isolated, so that the Conclusion would follow from Lemma 17.[31] Moreover, Lemma 14 tells us that if the Conclusion is false, there is $S^* < \infty$, defined as the supremum of all asset levels S greater than A_* such that $X(A) > A$ for all $A \in (A_*, S)$. It is easy to see that $X(S^*) = S^*$, so that in particular,

[31]By definition of A_*, there is $\{A'_n\}$ converging down to A_* with $X(A'_n) > A'_n$. If the assertion in the text is false, there is $\{A''_n\}$ also converging down to A_* along which $X(A''_n) \leq A'_n$. But then, using the fact that X is nondecreasing, there must be a third sequence along which equality holds, which brings us back to the case in which there are non-isolated sustainable assets.

S^* is sustainable. (We also note that $X(A^*) > A^*$, otherwise the Conclusion would be implied by setting $S_1 = A_*$ and $S_2 = S^*$, and applying Lemma 15.)

In the rest of the proof, then, we make the assumption (by way of ultimate contradiction) that the Conclusion is false. In particular, the construction above will be assumed to be valid. Also note that because many of the steps to follow presume that the Conclusion is false, they cannot all be regarded as relationships that truly hold in the model.

LEMMA **18.** *There are numbers $\epsilon > 0$, $\zeta > 0$ and $\eta > 0$ such that for every $A \in [S^*, S^* + \epsilon]$, there is an equilibrium which involves first-period continuation asset $A_1 \leq S^* - \zeta$, and has value $V(A) \leq V^s(S^*) - \eta$.*

Proof. By Lemma 12, there are $\zeta > 0$ and $\epsilon_1 > 0$ such that for every $A \in [S^*, S^* + \epsilon_1]$,

$$u\left(A - \frac{S^* - \zeta}{\alpha}\right) + \beta\delta V^s(S^* - \zeta) \geq u\left(A - \frac{A_1}{\alpha}\right) + \beta\delta V^s(A_1) \tag{47}$$

whenever $A_1 \geq S^*$, while at the same time (using the definition of S^*),

$$X(A'') > A'' \text{ for all } A'' \in [S^* - \zeta, S^*). \tag{48}$$

By part (i) of this proposition, there is $A_1 > B$ such that every equilibrium from $A \in [B, A_1)$ monotonically descends to B. By Lemma 7 (a), there must be a corresponding equilibrium which monotonically descends from A to S^* for every $A \in [S^*, \mu A_1)$, where $\mu = S^*/B$. Define $\epsilon_2 \equiv \min\{\epsilon_1, \mu A_1 - S^*\}$.

Using the first inequality in (29) of Lemma 8,

$$V^s(S^*) \geq \sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right) + u'\left(\frac{\alpha - 1}{\alpha} S^*\right)\left(\delta - \frac{1}{\alpha}\right)\zeta$$

for any path $\{A_t\}$ with the property that $A_t \leq S^*$ for all $t \geq 0$, and $A_1 \leq S^* - \zeta$. But then there exists $\eta > 0$ and $\epsilon_3 > 0$ such that

$$V^s(S^*) \geq \sum_{t=0}^{\infty} \delta^t u\left(A_t - \frac{A_{t+1}}{\alpha}\right) + \eta \tag{49}$$

for any path $\{A_t\}$ with $A_t \leq S^*$ for all $t \geq 1$, $A_1 \leq S^* - \zeta$, and $A_0 \leq S^* + \epsilon_3$. Define $\epsilon \equiv \min\{\epsilon_2, \epsilon_3\}$.

Pick any $A \in [S^*, S^* + \epsilon]$, and consider any "descending equilibrium" as described in the previous paragraph, with payoff $P(A)$. Suppose that it has continuation (A_1, V_1).

By Lemma 8, we know that $V_1 \leq V^s(A_1)$, so

$$u\left(A - \frac{A_1}{\alpha}\right) + \beta\delta V^s(A_1) \geq P(A). \tag{50}$$

Combining (47) and (50), we must conclude that

$$u\left(A - \frac{S^* - \zeta}{\alpha}\right) + \beta\delta V^s(S^* - \zeta) \geq P(A). \tag{51}$$

Now observe that (48), coupled with Lemma 14, implies that $H(S^* - \zeta) \geq V^s(S^* - \zeta)$. Using this information in (51), we must conclude that

$$u\left(A - \frac{S^* - \zeta}{\alpha}\right) + \beta\delta H(S^* - \zeta) \geq P(A). \tag{52}$$

It follows that the continuation $\{S^* - \zeta, H(S^* - \zeta)\}$ is an equilibrium specification from every $A \in [S^*, S^* + \epsilon]$.

To complete the proof, note that any path $\{A_t\}$ associated with this equilibrium continuation satisfies $A_t \leq S^*$ for all $t \geq 1$,[32] $A_1 \leq S^* - \zeta$, and $A_0 \leq S^* + \epsilon \leq S^* + \epsilon_3$. Therefore (49) applies. ■

LEMMA **19.** *Suppose that the Conclusion is false. Then*

(a) $d^*(S^*) < S^*$*, and* $d^*(A) < A$ *for every* $A \in (B, S^*)$ *with* $X(A) \neq A$*, and*

(b) $d(A) \leq A$ *for all* $A \in [B, S^*]$.

Proof. Part (a). First we show that $d^*(S^*) < S^*$. Suppose not; then, since $X(S^*) = S^*$, it follows from Lemma 3 that $d(S^*) = S^*$. By Lemma 18,

$$\hat{L}(S^*) < V^s(S^*).$$

Invoking (18) along with $d(S^*) = S^*$, we must conclude that

$$D(S^*) = u\left(\frac{\alpha - 1}{\alpha}S^*\right) + \beta\delta\hat{L}(S^*) < u\left(\frac{\alpha - 1}{\alpha}S^*\right) + \beta\delta V^s(S^*) = P^s(S^*),$$

or that $E(S^*) = P^s(S^*) - D(S^*) > 0$. By Lemma 16, the Conclusion must follow, a contradiction.

Next, consider any asset A with $X(A) < A$. By Lemma 3, $d^*(A) \leq X(A) < A$.

[32] This follows from $X(S^*) = S^*$ and the fact that X is nondecreasing.

Finally, consider $A \in (B, S^*)$ with $X(A) > A$. Suppose that $d^*(A) \geq A$ for some such A. Then, given that d^* is nondecreasing and usc, and that $d(S^*) < S^*$, there is a *maximal* asset $S \in (B, S^*)$ with $d^*(S) \geq S$ and $X(S) > S$. In fact, $d^*(S) = S$. We claim that

$$\hat{L}(S) \leq V^s(S). \tag{53}$$

By Lemma 5, L is increasing. So there is a sequence $\{A_n\}$ with $A_n \downarrow S$ and $L(A_n) = \hat{L}(A_n)$, with this common value converging to $\hat{L}(S)$. For each n, consider an equilibrium with the lowest value $V(A_n)$ among those that implement $Y(A_n)$.[33] Then

$$(1-\beta)u\left(A_n - \frac{Y(A_n)}{\alpha}\right) + \beta V(A_n) \geq D(A_n), \tag{54}$$

for all n. If strict inequality holds along a subsequence of n, then it's easy to see that $V(A_n) = u(A_n - B/\alpha) + \delta L(B)$ along that subsequence,[34] but $V(A_n)$ bounds $L(A_n)$ above, so that passing to the limit, (53) must certainly hold by Lemma 1. Therefore we may presume that for all n,

$$(1-\beta)u\left(A_n - \frac{Y(A_n)}{\alpha}\right) + \beta V(A_n) = D(A_n). \tag{55}$$

But in turn, we have that

$$D(A_n) = u\left(A_n - \frac{d^*(A_n)}{\alpha}\right) + \beta\delta\hat{L}(d^*(A_n)). \tag{56}$$

Combining (55) and (56), we see that for every n,

$$(1-\beta)u\left(A_n - \frac{Y(A_n)}{\alpha}\right) + \beta V(A_n) = u\left(A_n - \frac{d^*(A_n)}{\alpha}\right) + \beta\delta\hat{L}(d^*(A_n)). \tag{57}$$

Now we pass to the limit in (57). Recall that $\hat{L}(d^*(A_n))$ converges to $\hat{L}(S)$ (as does $L(A_n)$) and because d^* is usc and $A_n \geq S$ for all n, $d^*(A_n)$ converges to $d^*(S) = S$. Letting (Y, V) denote any limit point of $\{Y(A_n), V(A_n)\}$, we therefore have

$$(1-\beta)u\left(S - \frac{Y}{\alpha}\right) + \beta V = u\left(\frac{\alpha-1}{\alpha}S\right) + \beta\delta\hat{L}(S). \tag{58}$$

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[33]One can actually show that this value equals $L(A_n)$, but we do not use this fact anywhere in the proofs.

[34]We know that $Y(A_n)$ can be implemented by the continuation value $H(Y(A_n))$, satisfying (30). If strict inequality holds in (30), reduce continuation assets, always using H as the continuation, and sliding down the vertical portion of H at any point of discontinuity. (Public randomization allows us to do this.) Note that payoffs and continuation values change continuously as we do this. Eventually we come to $Y(A_n) = B$ with continuation value $L(B)$.

Transposing terms in (58), we establish (53) by noting that

$$\begin{aligned}\beta(1-\delta)\hat{L}(S) &\leq \beta V - \beta\delta\hat{L}(S)\\ &= u\left(\frac{\alpha-1}{\alpha}S\right) - (1-\beta)u\left(S - \frac{Y}{\alpha}\right)\\ &\leq u\left(\frac{\alpha-1}{\alpha}S\right) - (1-\beta)u\left(\frac{\alpha-1}{\alpha}S\right) = \beta u\left(\frac{\alpha-1}{\alpha}S\right), \qquad (59)\end{aligned}$$

where the first inequality uses $V(A_n) \geq L(A_n)$ for all n, so that $V \geq \hat{L}(S)$, and the second inequality uses $d^*(A_n) \geq Y(A_n)$ for all n, and $d^*(A_n) \to S$, so that $S \geq Y$.

With (53) in hand, we must conclude that

$$u\left(\frac{\alpha-1}{\alpha}S\right) + \frac{\beta\delta}{1-\delta}u\left(\frac{\alpha-1}{\alpha}S\right) \geq u\left(\frac{\alpha-1}{\alpha}S\right) + \beta\delta\hat{L}(S) = D(S), \qquad (60)$$

which means that S is sustainable. But $X(A) > A$ for all $A \in (S, S^*)$, so the conditions of Lemma 15 are satisfied. This means that the Conclusion holds, a contradiction.

Part (b). If false, then $d^*(A) > A$ for some $A \in [B, S^*]$. By part (a), $A < S^*$. Because d^* is nondecreasing (Lemma 4), $d(A') \geq A'$ over an interval to the right of A. But by part (a), we must have $X(A') = A'$ for all such A. That contradicts Lemma 17. ■

We now add a final element to our region between B and S^*. Define $S_* \geq B$ to be the largest asset level smaller than S^* such that $d^*(S_*) = S_*$. This is a well-defined object, because d^* is usc (Lemma 4), $d^*(A) \leq A$ for all $A \in [B, S^*]$ and $d^*(S^*) < S^*$ (Lemma 19), and $d^*(B) = B$ by part (i) of the proposition.[35] We record

LEMMA **20.** *$B \leq S_* < S*$, and $X(S_*) = S_*$.*

Proof. The assertion follows immediately from Lemma 19 coupled with $d^*(S_*) = S_*$. ■

Figure 7 summarizes the construction as well as the properties in Lemma 20. Panel A illustrates a case in which $S_* > B$, and Panel B, a case in which $S_* = B$. (Note: it is possible that $X(A) = A$ to the right of S^*, though by Lemma 17, this can only happen at isolated points.)

[35]Let S_* be the supremum over S in $[B, S^*]$ with $d^*(S) = S$ (well defined, because $d^*(B) = B$). Take any increasing sequence $\{S_n\}$ in $[B, S^*]$, $S^n \uparrow S_*$, with $d^*(S_n) = S_n$ for all n. Because d^* is usc, we have $d^*(S_*) \geq S_*$, and because $d^*(S_*) \leq S_*$ (part (b) of Lemma 19), it must be that $d^*(S_*) = S_*$. Note that by part (a) of Lemma 19, $S_* < S^*$.

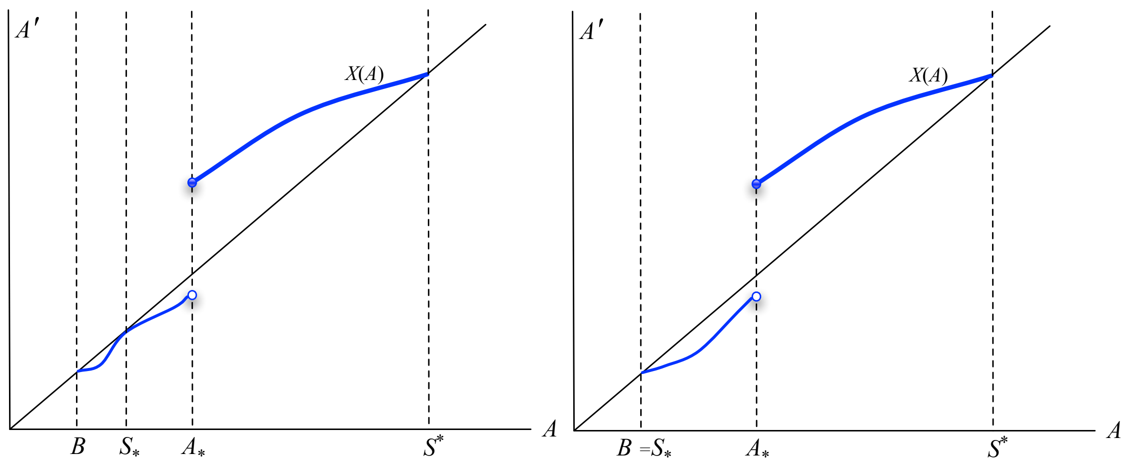

FIGURE 7. THE TWO SUSTAINABLE ASSETS S_* AND S^*.

Define $Y^+(A)$ as the limit of $Y(A_n)$ as A_n converges down to A. Given Lemma 9, $Y^+(A)$ is well-defined and $Y^+(A) \geq A$.

LEMMA **21.** *If the Conclusion is false,* $Y^+(S_*) \geq S_*$.

Proof. If $S_* = B$ the result is trivially true, so assume that $S_* > B$. Suppose, on the contrary, that $Y^+(S_*) < S_*$. We first establish a stronger version of (53); namely, that

$$\hat{L}(S_*) < V^s(S_*). \tag{61}$$

To this end, we carry out exactly the same argument as in the proof of part (a) of Lemma 19 leading to (58), with S_* in place of S.[36] Now observe that (59) — again, with S_* in place of S — must hold with strict inequality, because $S_* > Y^+(S_*) \geq Y$. We must therefore conclude that (60) holds with strict inequality, or that $E(S_*) > 0$. But then Lemma 16 assures us that the Conclusion must follow, which is a contradiction. ■

Let $\mu \equiv S^*/B$, and $\rho \equiv S_*/B$. Clearly, $\mu > \rho \geq 1$. Let $S_{**} \equiv \mu S_*$, and $S^{**} \equiv \mu S^*$. Note that $S_{**} = \mu S_* = \rho S^*$, so it can also be viewed as a scaling of S^* by the factor ρ. (Recall that by Lemmas 10 and 20, S_* is also sustainable, so Lemma 7 will apply with both the scalings μ and ρ.)

[36]Note again that equality must hold in (54). If strict inequality holds along a subsequence, then we know that $Y(A_n) = B$ and continuation values equal $L(B)$ along that subsequence, so that the stronger form (61) holds.

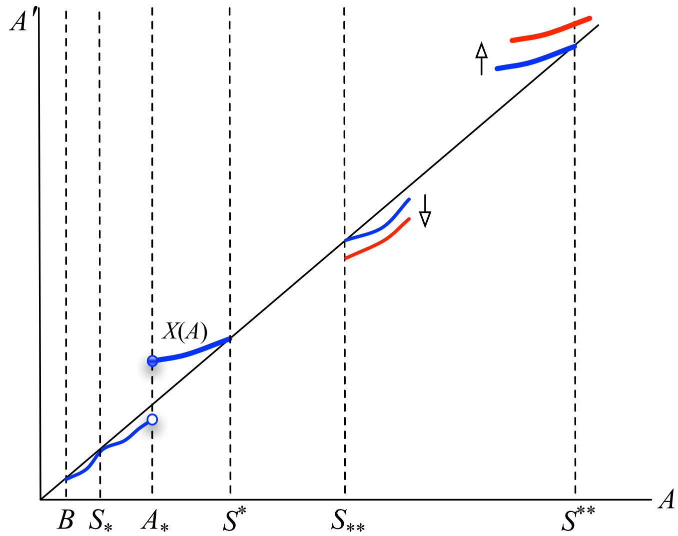

FIGURE 8. OUTLINE OF THE PROOF STARTING FROM LEMMA 22.

Here is an outline of the remainder of the proof. Use Figure 8. By Lemma 7 (a), equilibria at assets to the right of S_* and to the left of S^* can be "scaled up", using the factor μ. These are shown by the upper line to the right of S_{**} and the lower line to the left of S^{**}. But S_{**} is also a scaling of S^* (using the factor ρ), so we can use Lemma 7 again with Lemma 18 to achieve equilibria with even lower values (and lower continuation assets); see the lower segment to the right of S_{**}. Such equilibria support, in turn, *higher* asset values near S^{**}; see the upper line around S^{**}. The proof is completed by appealing to Lemma 16, which contradicts the starting presumption of this entire construction: that the Conclusion is false.

LEMMA **22.** (a) *For all* $A \geq B$,

(62) $$\hat{L}(\mu A) \leq \mu^{1-\sigma}\hat{L}(A).$$

(b) *For each* $\zeta > 0$*, there exists* $\eta > 0$ *such that for every* $A \in (S_*, S^*]$*, whenever* $Y(\mu A) \leq S_{**} - \zeta$,

(63) $$\hat{L}(\mu A') \leq \mu^{1-\sigma}\hat{L}(A') - \eta$$

for every $A' \in [S_*, A)$.

Proof. It is easy to see that Lemma 7 (a) implies (62). To prove part (b), fix $\zeta > 0$ and choose A as described, with $Y(\mu A) \leq S_{**} - \zeta$. Because $Y^+(S_*) \geq S_*$, any equilibrium that implements $L(A)$ has a continuation $\{A_1, V_1\}$ with $A_1 \geq S_*$. By Lemma 7 (a), $\{\mu A_1, \mu^{1-\sigma} V_1\}$ is an equilibrium continuation at $A'' \equiv \mu A > S_{**}$. So

$$u\left(A'' - \frac{\mu A_1}{\alpha}\right) + \beta\delta\mu^{1-\sigma}V_1 \geq D(A''), \tag{64}$$

and

$$\mu A_1 \geq \mu S_* = S_{**}. \tag{65}$$

Now, $Y(A'') \leq S_{**} - \zeta$ by assumption. Consider an equilibrium with the lowest continuation value $V(A'')$ among those that implement $Y(A'')$ from A''. Then

$$u\left(A'' - \frac{Y(A'')}{\alpha}\right) + \beta\delta V(A'') \geq D(A''). \tag{66}$$

If (66) does not bind, then we know that $Y(A'') = B$ and $V(A'') = L(B)$ (see footnote 34). Recalling that $A'' = \mu A$ and applying Lemma 1, we must therefore have

$$\begin{aligned} L(\mu A) &= u\left(\mu A - \frac{B}{\alpha}\right) + \delta L(B) \\ &\leq u\left(\mu A - \frac{\mu A_1}{\alpha}\right) + \delta\mu^{1-\sigma}V_1 - \frac{1-\beta}{\alpha\beta}u'\left(\mu A - \frac{B}{\alpha}\right)(\mu A_1 - B) \\ &\leq u\left(\mu A - \frac{\mu A_1}{\alpha}\right) + \delta\mu^{1-\sigma}V_1 - \frac{1-\beta}{\alpha\beta}u'\left(S^{**} - \frac{B}{\alpha}\right)(S_{**} - B) \\ &= \mu^{1-\sigma}L(A) - \frac{1-\beta}{\alpha\beta}u'\left(S^{**} - \frac{B}{\alpha}\right)(S_{**} - B), \end{aligned} \tag{67}$$

where the first inequality uses (15) of Lemma 1, and the second invokes (65) and $\mu A \leq S^{**}$. On the other hand, if (66) does bind, then using (64) and noting that $A'' = \mu A$,

$$u\left(\mu A - \frac{\mu A_1}{\alpha}\right) + \beta\delta\mu^{1-\sigma}V_1 \geq u\left(\mu A - \frac{Y(\mu A)}{\alpha}\right) + \beta\delta V(\mu A). \tag{68}$$

At the same time, by (65), $Y(\mu A) \leq S_{**} - \zeta \leq \mu A_1 - \zeta$ for all A satisfying the condition of part (b). Using this information in (68) and observing that $\mu A \leq S^{**}$,[37] we must conclude that there exists $\eta' > 0$ with $V_1 \geq V' + \eta_1$, where η_1 can be chosen independently of A. Therefore, using (68) again, there is $\eta_2 > 0$ such that

$$u\left(A'' - \frac{\mu A_1}{\alpha}\right) + \delta\mu^{1-\sigma}V_1 \geq u\left(A'' - \frac{Y(A'')}{\alpha}\right) + \delta V(A'') + \eta_2,$$

[37] This is needed to place a uniform lower bound on utility differences in (68).

or equivalently, $\mu^{1-\sigma}L(A) \geq L(\mu A) + \eta_2$. Combining this inequality with (67), and defining $\eta \equiv \min\{\eta_2, [(1-\beta)/\alpha\beta]u'(S^{**} - B/\alpha)(S_{**} - B)\}$, we have

$$\mu^{1-\sigma}L(A) \geq L(\mu A) + \eta \tag{69}$$

for all A satisfying the conditions of part (b). By Lemma 9, Y is nondecreasing, so $Y(\mu A') \leq Y(\mu A) \leq S_{**} - \zeta$ for every $A' \in (S_*, A]$, so (69) holds for all such A'. Because $\hat{L}$ is usc and nondecreasing (Lemma 5), (69) must extend to $\hat{L}$ over all $A' \in (S_*, A)$. By the right-continuity of $\hat{L}$ (Lemma 6), (69) must also apply to S_*. ■

LEMMA **23.** *$\hat{L}(\mu A) < \mu^{1-\sigma}\hat{L}(A)$ for all $A \in [S_*, S^*]$.*

Proof. By Lemma 18, there are $\epsilon' > 0$ and $\zeta' > 0$ such that for every $A' \in (S^*, S^* + \epsilon']$, $Y(A') \leq S^* - \zeta'$. Because $S_{**} = \rho S^*$, Lemma 7 (a) implies that $Y(\rho A') \leq S_{**} - \zeta$ for all such A', where $\zeta \equiv \rho\zeta'$. "Downscale" assets of the form $\rho A'$ by dividing by μ. Defining $\epsilon \equiv \rho\epsilon'/\mu$, it follows that part (b) of Lemma 22 applies to all $A \in [S_*, S_* + \epsilon]$.

Suppose, by way of contradiction, that $\hat{L}(\mu A) = \mu^{1-\sigma}\hat{L}(A)$ for some $A \in [S_*, S^*]$. Let A^* be the infimum over such A. Then $A^* \geq S_* + \epsilon$ (by the previous argument), and by the right-continuity of $\hat{L}$ (Lemma 6),

$$\hat{L}(\mu A^*) = \mu^{1-\sigma}\hat{L}(A^*). \tag{70}$$

Define $A' \equiv \mu A^*$ and observe that

$$\begin{aligned} D(\mu A^*) = D(A') &= u\left(A' - \frac{d^*(A')}{\alpha}\right) + \beta\delta\hat{L}(d^*(A')) \\ &= \mu^{1-\sigma}u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta\hat{L}\left(d^*(A')\right), \end{aligned} \tag{71}$$

where the second equality uses the constant-elasticity form of u. There are now two cases to consider. First, if $d^*(A')/\mu > d^*(A^*)$, then (71) implies

$$\begin{aligned} D(\mu A^*) &= \mu^{1-\sigma}u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta\hat{L}\left(d^*(A')\right) \\ &\leq \mu^{1-\sigma}\left[u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta\hat{L}\left(\frac{d^*(A')}{\mu}\right)\right] \\ &< \mu^{1-\sigma}D(A^*), \end{aligned} \tag{72}$$

where the weak inequality follows from (62), and the strict inequality from the fact that $d^*(A^*)$ is the *largest* maximizer of $u\left(A^* - x/\alpha\right) + \beta\delta\hat{L}\left(x\right)$, and $d^*(A')/\mu > d^*(A^*)$.

Otherwise, $d^*(A')/\mu \le d^*(A^*)$. By part (b) of Lemma 22 along with the definition of A^*, $d^*(A') \ge Y(A') = Y(\mu A^*) \ge S_{**}$, so that

$$S_* \le d^*(A')/\mu \le d^*(A^*) < A^*, \tag{73}$$

the last inequality following from the fact that $A^* > S_*$, while S_* is the largest value of $A \in [S_*, S^*]$ with $d^*(A) = A$. But then by part (a) of Lemma 22 and the definition of A^*, $\hat{L}(d^*(A')) < \mu^{1-\sigma}\hat{L}(d^*(A')/\mu)$. Using this inequality along with (71),

$$\begin{aligned} D(\mu A^*) &= \mu^{1-\sigma} u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta\hat{L}(d^*(A')) \\ &< \mu^{1-\sigma}\left[u\left(A^* - \frac{d^*(A')}{\mu\alpha}\right) + \beta\delta\hat{L}\left(\frac{d^*(A')}{\mu}\right)\right] \\ &\le \mu^{1-\sigma}\left[u\left(A^* - \frac{d^*(A^*)}{\alpha}\right) + \beta\delta\hat{L}(d^*(A^*))\right] \\ &= \mu^{1-\sigma} D(A^*), \end{aligned} \tag{74}$$

where the weak inequality above follows from the definition of $d^*(A^*)$. (72) and (74) together show that in all situations,

$$D(\mu A^*) < \mu^{1-\sigma} D(A^*). \tag{75}$$

Let $\{A_1, V_1\}$ be the equilibrium continuation that implements $L(A^*)$. By Lemma 7 (a), $\{\mu A_1, \mu^{1-\sigma}V_1\}$ is an equilibrium at μA^*, it has value equal to $\mu^{1-\sigma}L(A^*)$, and moreover, by the incentive constraint for $\{A_1, V_1\}$ coupled with (75),

$$u\left(\mu A^* - \frac{\mu A_1}{\alpha}\right) + \beta\delta\mu^{1-\sigma}V_1 \ge \mu^{1-\sigma}D(A^*) > D(\mu A^*).$$

This strict inequality, along with the fact that $\mu A_1 > B$, proves that one can lower equilibrium value at μA beyond the value created by scaling $\{A_1, V_1\}$, which shows that

$$L(\mu A^*) < \mu^{1-\sigma} L(A^*).$$

This contradicts the definition of A^*, and so completes the proof. ■

Proof of Proposition 4, part (ii). Assume the Conclusion is false. We claim that

$$E(S^{**}) = P^s(S^{**}) - D(S^{**}) > 0. \tag{76}$$

There are two possibilities to consider. First, $d^*(S^{**})/\mu \geq S_*$. In this case, the same argument as the one leading from (71) to (75) works to show that

$$D(S^{**}) < \mu^{1-\sigma} D(S^*). \tag{77}$$

(Replace A^* by S^* and A' by $\mu S^* = S^{**}$ in that argument. Our inequality $d^*(S^{**})/\mu \geq S_*$ is used to guarantee that (73) holds.) Because $P^s(S^{**}) = \mu^{1-s} P^m(S^*)$ and $P^m(S^*) \geq D(S^*)$, (77) immediately implies (76).

Otherwise, $d^*(S^{**})/\mu < S_*$. However, it must be the case that

$$d^*(S^{**})/\mu \geq B. \tag{78}$$

To see this, apply part (b) of Lemma 7 by setting the path $\{\mu A_t^*\}$ in that lemma to the constant path with asset level $S^{**} = \mu S^*$ at every date.[38] It follows from (78) that $A_1 \equiv d^*(S^{**})/\mu$ is a feasible asset choice at S^*.

Now, let d be a generic continuation asset choice that solves (18) at S^*. By Lemma 4 and the fact that $d^*(S_*) = S_*$, it must be the case that $d \geq S_*$. Because S^* is sustainable and $d \geq S_* > A_1 = d^*(S^{**})/\mu \geq B$,

$$P^s(S^*) \geq D(S^*) > u\left(S^* - \frac{A_1}{\alpha}\right) + \beta\delta \hat{L}(A_1). \tag{79}$$

Noting that $S^{**} = \mu S^*$ and $d^*(S^{**}) = \mu A_1$, we must conclude that

$$\begin{aligned} P^s(S^{**}) = \mu^{1-\sigma} P^s(S^*) &> \mu^{1-\sigma}\left[u\left(S^* - \frac{A_1}{\alpha}\right) + \beta\delta \hat{L}(A_1)\right] \\ &= u\left(S^{**} - \frac{d^*(S^{**})}{\alpha}\right) + \beta\delta\mu^{1-\sigma} \hat{L}(A_1) \\ &\geq u\left(S^{**} - \frac{d^*(S^{**})}{\alpha}\right) + \beta\delta \hat{L}(d^*(S^{**})) = D(S^{**}), \end{aligned}$$

where the first inequality uses (79) and the second inequality uses (62). That gives us (76) again.

By Lemma 16, this immediately precipitates a contradiction, because (76) implies that the Conclusion follows, while we have been working with the presumption that the Conclusion is false. ■

[38]This is our only use of part (b) of Lemma 7.

REFERENCES

Aleem, I. 1993. "Imperfect Information, Screening, and the Costs of Informal Lending: A Study of a Rural Credit Market in Pakistan." In K. Hoff, A. Braverman and J. Stiglitz (eds.), *The Economics of Rural Organization: Theory, Practice and Policy*, chap. 7. London: Oxford University Press (for the World Bank).

Ambec and Treich

Anderson and baland

Appadurai

Goldstein

Udry

Banerjee

Duflo

de Mel

Banerjee and Duflo

Banerjee and Newman

Bénabou and Tirole

Benhabib and Bisin

Bernheim Ray Yeltekin

Shipton

Benartzi

Ainslie 1975, 1992

Ashraf

Akerlof

Aliber

Gugerty 2001 and 2007

Abreu

Basu, K. 2011. “Hyperbolic Discounting and the Sustainability of Rotational Savings Arrangements.” *American Economic Journal: Microeconomics* **3**, 143–71.

Khera, R. 2011. “India’s Public Distribution System: Utilisation and Impact.” *Journal of Development Studies* **47**, 1038–1060.

Thaler (1992)

Laibson 1994 1997 1997

O’Donoghue

Ray, D. 2006. “Aspirations, Poverty and Economic Change.” In A. Banerjee, R. Bènabou, and D. Mookherjee (eds), *What Have We Learnt About Poverty*, Oxford: Oxford University Press.

Genicot, G. and D. Ray 2012. “Aspirations and Inequality.” Mimeo., Department of Economics, new York University.

Strotz

Phelps and Pollak