
GRADUATE SCHOOL OF BUSINESS

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# A New LMM

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# Background

- About LMM
  - pros: convenient
    - use of the Black-Scholes model
    - forward measure
  - cons: problems
    - drift freezing
    - forward LIBOR is no longer log-normal
    - no continuous limit (continuous forward rate cannot be log-normal)

# Background

- Forward measure
  - maturity dependent
  - Jamshidian's separation theorem (1987)

$$\tilde{\mathbb{E}}_t\left[\exp\ -\int_t^T r(u)du\ \ X(T)\right] = \mathbb{E}_t\left[\exp\ -\int_t^T r(u)du\ \right]\tilde{\mathbb{E}}_t^{(T)}[X(T)]$$

$$= P(t,T)\tilde{\mathbb{E}}_t^{(T)}[X(T)]$$

# A Replacement

- 1+f (=F) follows log-normal
  - pros, only pros, no cons
  - consistent with normal short rate models
    - Vasicek and its variations such as
    - Hull-White
  - no need for drift freezing
    - forward measures are deterministically linked
  - same Black-Scholes model for cap/floor
    - call -> put (Hull)
  - price caps/floors and swaptions simultaneously!

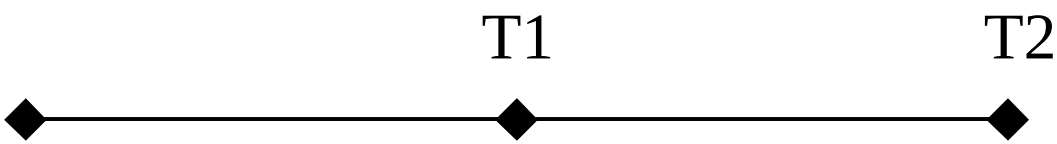
# A Replacement

- 1+f (=F) follows log-normal
  - closed-form solution to swaptions
    - Jamshidian's theorem (yet one-factor only)
    - premium quotes anyway (otherwise log-normal volatility would be a problem)
  - approach normal when f is low and maintain shifted log-normal when it is high
  - make perfect economic sense
    - F (forward rate) is reciprocal of $\Psi$ (forward price) which leads to a period-shift in forward measure

# The Math

- The Key is F (forward rate) is reciprocal of $\Psi$ (forward price) which leads to a period-shift in forward measure
  - F(t,Ti,Tj) is Tj-forward measure
  - $\Psi$(t,Ti,Tj), which is 1/F is Ti-forward measure
  - proof in paper
  - it makes perfect sense in that
    - payment on F is paid at Tj
    - but this payment can be discounted (since F is known at Ti) to Ti, which then becomes $\Psi$.

# The Visual




$\max\{L12 - R^K, 0\} =$
$\max\{F112 - R^K, 0\}$
where F112 is martingale under W^2

$(1/K) \max\{K - P12, 0\}$
$K = 1 + R^K$

Note that L12 (and F112) is known at time T1 but used at time T2.

# The Key Equation

$$d\tilde{W}^{(T_j)}(t) = d\tilde{W}^{(T_i)}(t) + \xi(t, T_i, T_j)dt$$

where

$$\xi(t, T_i, T_j) = v(r, t, T_j) - v(r, t, T_i)$$

$$d\ln P(r, t, T) = \quad r(t) - \frac{v(r,t,T)^2}{2} \quad dt + v(r, t, T)d\hat{W}(t)$$

which implies

$$\begin{aligned} \xi(t, T_i, T_j) &= \xi(t, T_i, T_k) + \xi(t, T_k, T_j) \\ &= \xi(t, T_i, T_{i+1}) + \cdots + \xi(t, T_{j-1}, T_j) \end{aligned}$$

# The Key Equation

$$\frac{dF(t,T_i,T_j)}{F(t,T_i,T_j)} = \xi(t,T_i,T_j)d\tilde{W}^{(T_j)}(t)$$

$$= \xi(t,T_i,T_j) \quad d\tilde{W}^{(T_i)}(t) + \xi(t,T_i,T_j)dt$$

$$= \xi(t,T_i,T_j)^2 + \xi(t,T_i,T_j)d\tilde{W}^{(T_i)}(t)$$

$$= \sum_{k=i}^{j-1} \xi(t,T_k,T_{k+1}) \quad ^2 + \sum_{k=i}^{j-1} \xi(t,T_i,T_j)d\tilde{W}^{(T_i)}(t)$$

# ξ (The Key Variable)

- instantaneous volatility under forward measure for forward rate
- deterministic
- caplet "put" vol is $\sqrt{\int_t^{T_j} \xi(u,T_j,T_{j+1})^2 du}$ which is just like the BS "price" vol
  - note call on rate = put on price

# ξ (The Key Variable)

- caplet (put) pricing formula

$$c_{P,j} = \frac{P(t,T_j)}{K} \; KN \left( \frac{\ln \Psi(t,T_j,T_{j+1}) - \ln K - \text{浇} v_{P,j}^2}{v_{P,j}} \right) - \Psi(t,T_j,T_{j+1}) N \left( \frac{\ln \Psi(t,T_j,T_{j+1}) - \ln K + \quad v_{P,j}^2}{v_{P,j}} \right)$$

where (price vol)

$$\begin{aligned} v_{P,j}^2 &= \tilde{\mathbb{V}}[\ln P(r,T_j,T_{j+1})] \\ &= \tilde{\mathbb{V}}[\ln \Psi(T_j,T_j,T_{j+1})] \\ &= \int_t^{T_j} \xi(u,T_j,T_{j+1})^2 \, du \end{aligned}$$

# Lognormal vol vs. Normal vol

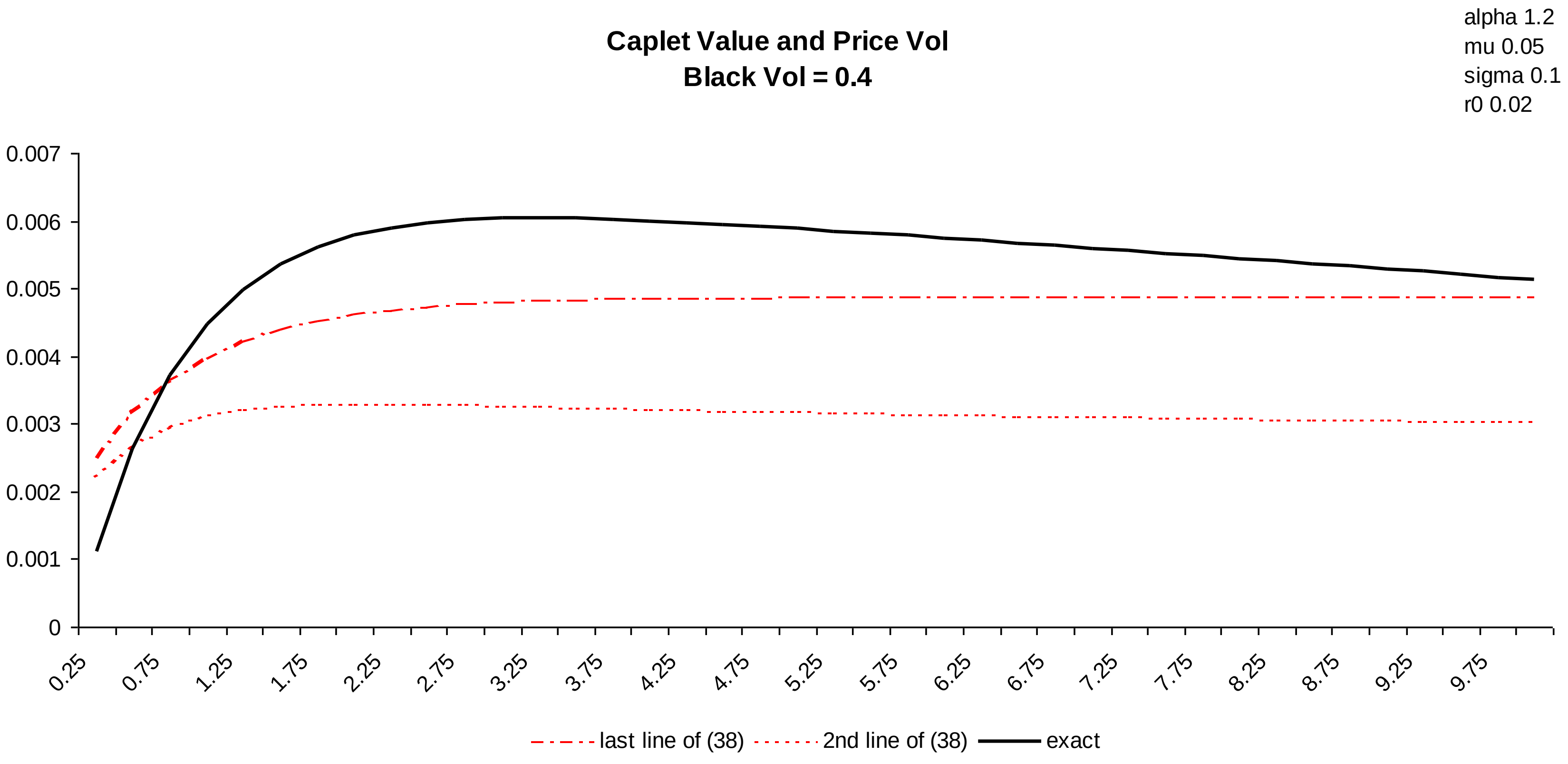
- volatility of discrete LIBOR rate

$$
\begin{aligned}
\tilde{\mathbb{V}}_t^{(T_{j+1})}[\ell(T_j, T_{j+1})] &= \tilde{\mathbb{V}}_t^{(T_{j+1})}[-\ln P(r, T_j, T_{j+1})] \\
&= \tilde{\mathbb{V}}_t^{(T_{j+1})}[\ln P(r, T_j, T_{j+1})] \\
&= \tilde{\mathbb{V}}_t^{(T_j)}[\ln P(r, T_j, T_{j+1})] \\
&= v_{P,j}^2
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\mathbb{V}}_t^{(T_{j+1})}[\ell(T_j, T_{j+1})] &= e^{2\mu + 2v_{\ell,j}^2} - e^{2\mu + v_{f,j}^2} \\
&= f(t, T_j, T_{j+1})^2 \quad e^{v_{f,j}^2} - 1 \\
&\approx f(t, T_j, T_{j+1})^2 (1 + v_{f,j}^2 - 1) \\
&= (f(t, T_j, T_{j+1}) v_{f,j})^2
\end{aligned}
$$

$$
v_{f,j}^2 = \tilde{\mathbb{V}}_t^{(T_{j+1})}[\ln \ell^*(T_j, T_{j+1})]
$$

# Lognormal vol vs. Normal vol



# Swaption

- SMM
  - swap rate (w)

$$w(t,\underline{T}) = \frac{\sum_{j=1}^{n} P(r,t,T_j) f(t,T_j,T_{j+1})}{\sum_{j=1}^{n} P(r,t,T_j)} = \frac{1 - P(r,t,T_n)}{\sum_{j=1}^{n} P(r,t,T_j)}$$

  - swaption

$$\begin{aligned} c_{w,j,n} &= \hat{\mathbb{E}}_t \left[ \exp\ -\int_t^s r(u)\ \sum_{j=1}^{n} P(r,s,T_j) \max\{w(s) - w_K, 0\} \right] \\ &= P(t,s) \tilde{\mathbb{E}}_t^{(s)} \left[ \sum_{j=1}^{n} P(r,s,T_j) \max\{w(s) - w_K, 0\} \right] \\ &= P(t,s) \tilde{\mathbb{E}}_t^{(s)} \left[ \sum_{j=1}^{n} P(r,s,T_j) \right] \tilde{\mathbb{E}}_t^{\Sigma}\ \max\{w(s) - w_K, 0\} \\ &= \sum_{j=1}^{n} P(r,t,T_j) \tilde{\mathbb{E}}_t^{\Sigma}\ \max\{w(s) - w_K, 0\} \end{aligned}$$

# Swaption

- SMM
  - SMM and LMM cannot co-exist (Jamshidian)
  - swap rate and LIBOR cannot be simultaneously log-normal
  - proof:

# Swaption

- $E^{\Sigma}[w(s)]=w(t)$ only if $n \rightarrow \infty$

$$\hat{\mathbb{E}}_t\left[\exp\left(-\int_t^s r(u)du\right)\frac{\sum_{j=1}^n P(r,s,T_j)}{\sum_{j=1}^n P(r,t,T_j)}w(s)\right]$$
$$=\hat{\mathbb{E}}_t\left[\exp\left(-\int_t^s r(u)du\right)\frac{\sum_{j=1}^n P(r,s,T_j)}{\sum_{j=1}^n P(r,t,T_j)}\right]\tilde{\mathbb{E}}_t^{\Sigma}[w(s)]$$
$$=\tilde{\mathbb{E}}_t^{\Sigma}[w(s)]$$

$$\hat{\mathbb{E}}_t\left[\exp\left(-\int_t^s r(u)du\right)\frac{\sum_{j=1}^n P(r,s,T_j)}{\sum_{j=1}^n P(r,t,T_j)}w(s)\right]$$
$$=w(t)-\frac{1-P(r,t,s)}{\sum_{j=1}^n P(r,t,T_j)}$$
$$\approx w(t)$$

# Swaption

- Pricing formula
  - swap P&L

$$V(u) = (w(u) - w(t))\sum_{j=1}^{n} P(r,u,T_j)$$
$$= \text{floating rate bond} \quad - \quad \text{fixed rate bond}$$
$$= 1 - B(u,\underline{T};w_t)$$

  - swaption payoff

$$\max\{w(u) - w_K, 0\}\sum_{j=1}^{n} P(r,u,T_j) = \max\{1 - B(u,\underline{T};w_K), 0\}$$

  - Jamshidian Theorem (breaking up K)

$$c_{B,j,n} = \max \quad w_K\sum_{j=1}^{n} K_j + K_n \quad - \quad w_K\sum_{j=1}^{n} P(r,u,T_j) + P(r,u,T_n) \;\; ,0$$
$$= \sum_{j=1}^{n} w_K \max \;\; K_j - P(r,u,T_j), 0 \;\; + \max\{K_n - P(r,u,T_n), 0\}$$

# Swaption

- Pricing formula

$$c_{B,j,n} = \sum_{j=1}^{n} w_K x_j + x_n$$

$$x_j = P(r,t,T_j)\ \ K_j N\ \frac{\ln \Psi(t,T_j,T_{j+1}) - \ln K - \text{浇} v_{P,j}^2}{v_{P,j}} - \Psi(t,T_j,T_{j+1}) N\ \frac{\ln \Psi(t,T_j,T_{j+1}) - \ln K + \ \ v_{P,j}^2}{v_{P,j}}$$

– swpation is price quote, not vol quote

– so no reason to do SMM

– however, not suitable for multiple factors

  - use it anyway as an approximation (Chen-Scott)

# The Perfect Calibration

- caplet is a special swaption: a into b
  - 1×1 swaption (=caplet)

$$\int_t^{T_1} \xi(u,T_1,T_2)^2\,du = \xi_{012}^2\Delta_1 = x_{11}^2$$

  - 1×2 swaption

$$\int_t^{T_1} \xi(u,T_1,T_2)^2\,du = \xi_{012}^2\Delta_1 = x_{11}^2$$

$$\int_t^{T_1} \xi(u,T_1,T_3)^2\,du = \int_t^{T_1} \{\xi(u,T_1,T_2) + \xi(u,T_2,T_3)\}^2\,du$$

$$= \{\xi_{012} + \xi_{023}\}^2\Delta_1$$

  - 2×1 swaption

$$\int_t^{T_2} \xi(u,T_2,T_3)^2\,du = \int_t^{T_1} \xi(u,T_2,T_3)^2\,du + \int_{T_1}^{T_2} \xi(u,T_2,T_3)^2\,du$$

$$= x_{12}^2 + x_{22}^2 = \xi_{023}^2\Delta_1 + \xi_{123}^2\Delta_2$$

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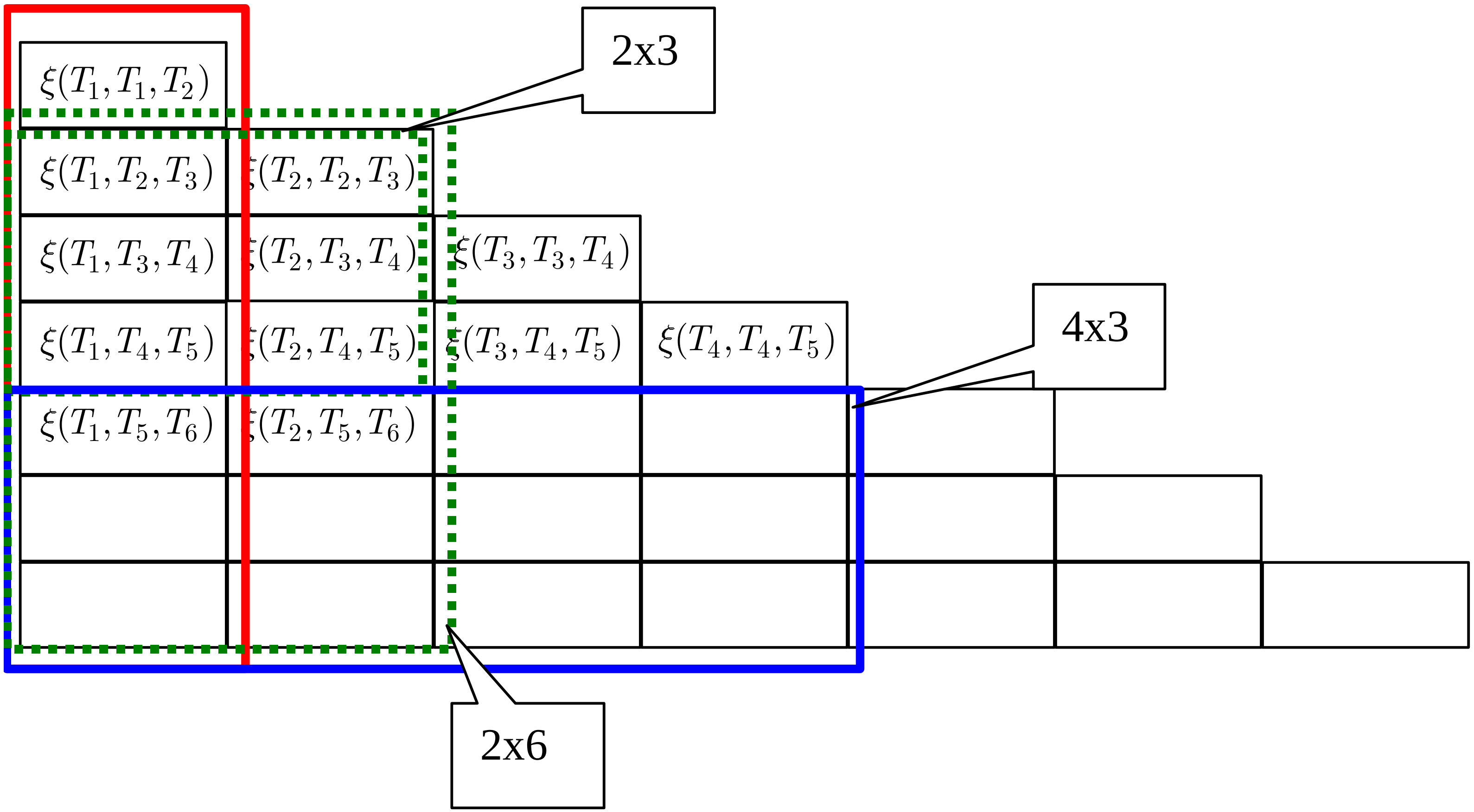
# The Perfect Calibration

- caplet is a special swaption: a into b

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# Simulation

- different from LMM
- no approximation
- simulate all forward measures at all times
  - okok… this is a con. but this is error-free (as opposed to LMM where errors accumulate over time) and as a result, our model can price VERY LONG swaptions and exotics
  - vendors (e.g. Dxxx and Nxxx) cannot price swaptions accurately over 10 years.

# Simulation

- Although measures are linked, but realistically for each discounting, only the corresponding (forward) measure can be used
- Hence, canNOT discount along path
  - Cash flows, yes.
  - Discounting can only be “outside” of MC

# Extensions

- Thanks to (log)normality, really straightforward
  - to multiple factors
    - e.g. CMS
  - to multiple curves
    - e.g. FX
  - no need to change any structure of single factor
  - a newly added layer (ideal for software engineering)

# Extensions

- Need to think about more flexibility between normal and log-normal
  - currently no flexibility
  - could lose model-free result

# Stochastic $\xi$

- Example
  - Vasicek (everything works out)

$$v_{P,j}^2 = \left( \frac{1 - e^{-\alpha(T_{j+1} - T_j)}}{\alpha} \right)^2 \sigma^2 \frac{1 - e^{-2\alpha(T_j - t)}}{2\alpha}$$

$$= \int_t^{T_j} \xi(u, T_j, T_{j+1})^2 \, du$$

    - for caplets only, piece-wise flat $\alpha$
  - Cox-Ingersoll-Ross (nothing works out)
    - volatility is not linear anymore
    - forward measures still valid yet not helpful
    - simulations still okay but no calibration

# Conclusion

- Solve LMM problem by letting 1+L be lognormal
- Exact drift adjustment
- Perfect calibration of caps and swaptions