

Optimal Investment and Pricing in the Presence of Defaults

Scott Robertson
Questrom School of Business, Boston University

Joint Work with Tetsuya Ishikawa
Morgan Stanley

ECMF
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Research Goals

Solve the optimal investment problem when the underlying traded asset may default.

- Price defaultable bonds.
- Price dynamic default insurance.

Obtain explicit answers.

- Provide a PDE counterpart to the BSDE pricing literature.

Motivation

Say our goal is to price a claim whose payoff is contingent upon survival of a reference entity.

- Payoff: $\phi_T 1_{T > \delta}$
- δ : default time of a firm S .

In practice, pricing is done under a risk neutral measure.

Two problems:

- What risk neutral measure?
- What is the underlying traded asset? What if the underlying is the reference entity?

Motivation

Say our goal is to insure ourselves against losses from the default of a stock in which we own a position.

We could enter into a CDS

- What if investment horizon does not match CDS maturity?
- What if we want dynamic protection?

Is there a fair price for dynamic protection taking into account market incompleteness, and our preferences?

Contribution to the Literature

Optimal investment and indifference pricing with defaults have been extensively studied.

- Primarily from the "BSDE" perspective, especially with respect to pricing.
- We fill in a gap by considering Markovian factor models, using PDE techniques, and focusing on indifference pricing.
 - Amenable to computation and analysis.

The computation of dynamic default insurance has been much less well-studied.

Contribution to the Literature

(selected) "PDE" articles

- [Lin06]: Merton model with default intensity $\gamma_t = \gamma(S_t)$ under a fixed risk neutral measure. Analytical formulas for European option prices.
- [SZ07]: single stock factor model similar to ours. However, investor does not lose money in stock upon default.
- [BBC16]: risk-sensitive control problem in factor model with multiple securities, default state dependent intensities. Investor does not lose money in stock upon default.
- [BC16]: optimal investment/consumption problem for power utility in a factor model with multiple securities, default state dependent intensities. Investor loses money upon default.

Contribution to the Literature

(selected) "BSDE" articles

- [Mor09, LQ11]: single stock and non-traded claim. Brownian setting prior to default.
- [JP11, JKP13]: single/multiple stocks along with claim. Multiple credit events which cause a jump in stock prices with trading possible after jump. Brownian setting
- [MS17]: stock modeled as a pure-jump Levy process.
- [LQ15]: extension of [LQ11] to partial information models.
- [GN15, CGN15]: mean-variance hedging under default risk.

Model

Reduced form, "hybrid" intensity model: [SZ07].

X : underlying factor process

- $dX_t = b(X_t)dt + a(X_t)dW_t$.
 - W : d -dim B.M.. $b, A := aa'$ smooth, A locally elliptic.
- Solution to Martingale problem for L on $E \subseteq \mathbb{R}^d$ where
 - $L = (1/2)\text{Tr}(AD^2) + b'\nabla$
 - $E = \cup_n E_n$ with E_n bounded, $E_n \uparrow$, ∂E_n smooth.

One risky asset S (riskless asset set to 1)

- S defaults at the random time δ . Prior to δ , S has instantaneous returns, variances, correlations driven by X .

Model

Start at $t \geq 0$. $X_t^{t,x} = x \in E$. Write $X = X^{t,x}$.

$$\frac{dS_s}{S_s} = 1_{s \leq \delta} \left((\mu - \gamma)(X_s) ds + (\sigma \rho) (X_s)' dW_s + \left(\sigma \sqrt{1 - \rho' \rho} \right) (X_s) dW_s^0 \right) - dM_s; \quad s \geq t.$$

- W^0 : one-dim B.M. $\perp\!\!\!\perp$ of W .
- $\delta := \inf \{s > t : \int_t^s \gamma(X_u) du = -\log(U)\}$, $U \perp\!\!\!\perp W, W^0$.
- $H_s := 1_{s \geq \delta}$; $M_s := H_s - \int_t^{s \wedge \delta} \gamma(X_u) du$,
- $\mathbb{G} := \mathbb{F}^{W, W^0} \vee \mathbb{F}^H$. W, W^0, M are \mathbb{G} local martingales.
- $\mu, \sigma, \gamma, \rho$ smooth functions on E , $\gamma, \sigma > 0$, $\rho' \rho \leq 1$.

Optimal Investment Problem

Investment horizon: $[t, T]$ for $T > t$.

\mathcal{M} : equivalent local martingale measures on \mathcal{G}_T . $\widetilde{\mathcal{M}}$ subset with finite relative entropy w.r.t. \mathbb{P} .

\mathcal{A} : acceptable (dollar) trading strategies π .

- Wealth process $\mathcal{W}^{\pi, w} = w + \int_t^\cdot \pi_u dS_u / S_{u-}$.
- Dollar position π_δ lost at δ .

$\pi \in \mathcal{A}$ if $\mathcal{W}^{\pi, w}$ is a \mathbb{Q} local martingale for all $\mathbb{Q} \in \widetilde{\mathcal{M}}$.

Optimal Investment Problem

Exponential investor: $U(w) := -e^{-\alpha w}$, $w \in \mathbb{R}$.

Investor

- Trades in S according to $\pi \in \mathcal{A}$.
- Owns a non-traded claim with time T payoff $\phi(X_T)1_{T < \delta}$.
- ϕ smooth, bounded. Primarily care about $\phi \equiv 1, \phi \equiv 0$.

For 0 initial wealth write $\mathcal{W}^\pi = \mathcal{W}^{\pi,0}$ and define

$$u(t, x; \phi) := \sup_{\pi \in \mathcal{A}} E \left[-e^{-\alpha(\mathcal{W}_T^\pi + \phi(X_T)1_{\delta > T})} \right]; \quad (X_t = x)$$
$$G(t, x; \phi) := -\frac{1}{\alpha} \log(-u(t, x; \phi)).$$

$G(t, x; \phi) = -\frac{1}{\alpha} \log(-u(t, x; \phi))$: Certainty Equivalent

Heuristics using DPP suggest G should solve

$$0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right);$$
$$\phi = G(T, \cdot)$$

- $\theta(y)$: inverse of ye^y and $\theta_G := \theta \left(\frac{\gamma}{\sigma^2} e^{\frac{\mu}{\sigma^2} + \alpha G - \frac{\alpha}{\sigma} \nabla G' a \rho} \right)$.

If G is a classical solution, DPP suggests optimal strategy is

- $\hat{\pi}_s = \hat{\pi}(s, X_s^{t,x})$ for $\hat{\pi} = \frac{1}{\alpha} \left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho - \theta_G \right)$.

Certainty Equivalent PDE

$$0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right);$$
$$\phi = G(T, \cdot)$$

- This is a semi-linear degenerate parabolic PDE.
 - Non-linearities arise due to market incompleteness.
- Luckily: $\theta(y) \approx \log(y) - \log(\log(y)), y \gg 0$.
 - PDE is quadratically growing in $G, \nabla G$.
- Regarding solutions/verification:
 - For general regions E , local ellipticity, verification is hard: lack gradient estimates near ∂E .
- We must enforce some additional (global) condition.

The Main Assumption

Set $\ell := (\mu - \gamma)/\sigma$ (market price of risk).

Today: assume "strictly incomplete" market absent default.

- The paper treats the "complete" case as well.

Main assumptions:

- $\sup_{x \in E} \rho' \rho(x) < 1$.
- For some $\varepsilon > 0$ we have for each n

$$\sup_{x \in \bar{E}_n} E^x \left[e^{\varepsilon \int_0^T \ell(X_u)^2 du} \right] = C(\varepsilon, n) < \infty.$$

This assumption is MILD. Holds in virtually all models.

- E.g. $X \sim OU, CIR, \mu, \sigma^2, \gamma$ affine.

The Main Result

$$0 = G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G + \frac{\sigma^2}{2\alpha} \left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right);$$

$$\phi = G(T, \cdot)$$

Theorem: assume $\sup_{x \in E} \rho' \rho(x) < 1$ and for some $\varepsilon > 0$:

$$\cdot \sup_{x \in \bar{E}_n} E^x \left[e^{\varepsilon \int_0^T \ell(X_u)^2 du} \right] = C(n) < \infty, \forall n.$$

Then

- The certainty equivalent G is a classical ($C^{1,2}$) solution.
- The optimal trading strategy is
 - $\hat{\pi}_s = \hat{\pi}(s, X_s^{t,x})$ for $\hat{\pi} = \frac{1}{\alpha} \left(\frac{\mu}{\sigma} - \frac{\alpha}{\sigma} \nabla G' a \rho - \theta_G \right)$.
- The optimal martingale measure $\hat{\mathbb{Q}}$ has density
 - $\hat{Z}_s = e^{-\alpha(\mathcal{W}_s^{\hat{\pi}} - G(t,x;\phi) + 1_{\delta > s} G(x, X_s; \phi))}$.

Application: Pricing for Defaultable Bonds

Investor owns q units notional: claim payoff $q1_{\delta > T}$.

(per-unit, buyer's) indifference price: $p(t, x; q)$ solving

- $u(t, x; 0, 0) = u(t, x; q, -qp(t, x; q)) = e^{\alpha qp(t, x; q)} u(t, x; q, 0)$.
- $u(t, x; \phi, w)$: utility for initial wealth w .
- Well known p does not depend on w .

Immediate result as $G(t, x; q) = -(1/\alpha) \log(-u(t, x; q))$:

- $p(t, x; q) = \frac{1}{q} (G(t, x; q) - G(t, x; 0))$.

Application: Dynamic Default Insurance

Goal: find a fair price for dynamic protection against default.

- Approximation to CDS pricing valid for frequent contract adjustments.

Motivation from [SZ07]: optimal investment/pricing but with no loss at default.

- π_δ not lost at default time δ .

How is this possible? What contract has been entered into which enables this?

Dynamic Default Insurance

Perspective: investor has two alternatives:

- A) Do not purchase protection. Lose π_δ at δ . Indirect utility of $u(t, x)$.
- B) Purchase protection. Pay a (per-unit) cash flow rate of f , where f is to-be-determined.
 - Wealth dynamics:

$$d\mathcal{W}_s^{\pi, d} = \pi_s 1_{s \leq \delta} ((\mu - \gamma)(X_s) - f_s) ds \\ + \pi_s 1_{s \leq \delta} \left((\sigma \rho)(X_s)' dW_s + (\sigma \sqrt{1 - \rho' \rho})(X_s) dW_s^0 \right).$$

- Indirect utility

$$u^d(t, x) := \sup_{\pi \in \mathcal{A}_d} E \left[-e^{-\alpha \mathcal{W}_T^{\pi, d}} \right].$$

Dynamic Default Insurance

$$G(t, x) = -\frac{1}{\alpha} \log(-u(t, x)); \quad G^d(t, x) = -\frac{1}{\alpha} \log(-u^d(t, x)).$$

Guess $f_t = f(t, X_t)$. Find f so that PDEs for G, G^d are the same (both have terminal condition ϕ).

$$\begin{aligned} 0 = & G_t + LG - \frac{\alpha}{2} \nabla G' A \nabla G \\ & + \frac{\sigma^2}{2\alpha} \left(\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} - \theta_G^2 - 2\theta_G \right); \\ 0 = & G_t^d + LG^d - \frac{\alpha}{2} \nabla (G^d)' A \nabla G^d \\ & + \frac{\sigma^2}{2\alpha} \left(\left(\frac{\mu - f}{\sigma^2} - \frac{\alpha}{\sigma} \nabla (G^d)' a \rho \right)^2 + \frac{2\gamma}{\sigma^2} (1 - e^{\alpha G^d}) \right). \end{aligned}$$

Dynamic Default Insurance

Upon inspection, given a solution G to the first PDE, G will solve the second PDE if f satisfies

$$\frac{f_{\pm}}{\sigma^2} = \frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \pm \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho\right)^2 - \left(\theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G}\right)}.$$

- Term inside square root is non-negative: real solutions.
- We choose the "-" solution.
 - Lowest possible f since this is what the investor pays.
 - Can also justify f_- by inspecting optimal strategies π_{\pm}^d : $f_+ > 0$ and $\pi_+^d < 0$ - not feasible.

Dynamic Default Insurance

We define the dynamic default insurance protection price

$$f := \sigma^2 \left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{\alpha}{\sigma} \nabla G' a \rho \right)^2 - \left(\theta_G^2 + 2\theta_G - \frac{2\gamma}{\sigma^2} e^{\alpha G} \right)} \right).$$

Facts

- $f \leq \gamma e^{\alpha(G+\hat{\pi})} = \gamma^{\hat{\mathbb{Q}}}$: the default intensity under the dual optimal measure $\hat{\mathbb{Q}}$.
 - Equality only when $\hat{\pi} = 0$.
- $f > 0$ when $\hat{\pi} > 0$: intuitive. Pay for protection when long.
- $f > 0$ possible even when $\hat{\pi} < 0$, but $f < 0$ for $\hat{\pi} \ll 0$.

Numerical Application

Application: $X \sim CIR$, affine market price of risk.

- $dX_t = \kappa(\theta - X_t)dt + \xi\sqrt{X_t}dW_t$.
- Prior to default
 - $dS_t/S_t = \mu X_t dt + \sigma\sqrt{X_t} \left(\rho dW_t + \sqrt{1 - \rho^2} dW_t^0 \right)$.
- Default intensity: $\gamma_t = \gamma X_t$.

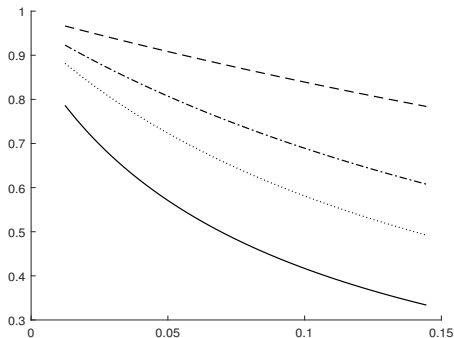
Assume $\mu \in \mathbb{R}, \sigma, \gamma > 0$ and $|\rho| < 1$.

- Main assumption holds provided $\kappa\theta > \xi^2/2$.

Application: Defaultable Bond Pricing

Investor owns q units of a defaultable bond.

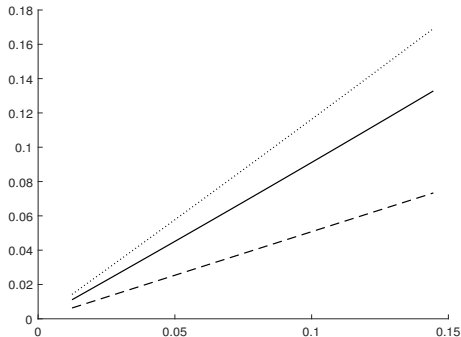
$p(0, x; q)$ as a function of q, x for $T = 1$.



- Physical default prob of 3% at $x = 6\%$ (long run mean).
- $q = 1$ (dash), $q = 3$ (dot-dash), $q = 5$ (dot), $q = 10$ (solid).









Application: Dynamic Default Insurance

$f(0, x)$ as a function of x for $T = 1$.



- $\gamma^{\hat{Q}}$ (dash), f (solid), γ (dash).

THANK YOU!

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