Cost Efficiency in Incomplete Markets

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The cost efficiency principle

Idea (Cox & Leland, Dybvig)

- Consider the utility maximization problem in a complete market

$$\sup_{X \in \mathcal{X}(x)} \mathbb{E}\big[U(X_T)\big] = \mathbb{E}\big[U(X^*)\big]$$

- Then, if the cdf of X^* is F, we have also that X^* is the minimizer of

$$\inf_{Y \sim F} \mathbb{E}\big[\xi Y\big] = \inf_{Y \sim F} c(Y)$$

for pricing kernel ξ and replication pricing functional $c(\cdot)$

The cost efficiency principle

- Moreover, if ξ has a continuous distribution,

$$X^* = F^{-1}\big(1 - F_\xi(\xi)\big)$$

by the Fréchet-Hoeffding bounds

The cost efficiency principle

- This generalizes to the more general setting (Bernard, Boyle, Vanduffel)

$$\sup_{X \in \mathcal{X}(x)} V(X_T) = V(X^*)$$

for $V : L^0 \to \mathbb{R}$ preserving first order stochastic dominance (or equivalently, V is law invariant and increasing)

Application 1

Distribution Builder

- Investors are notoriously bad in estimating their utility function
- Instead it has been suggested investors should specify there target distribution of the portfolio at terminal time that is reachable with given initial captial (Goldstein, Sharpe & Blythe; Goldstein Johnson and Sharpe; Monin)

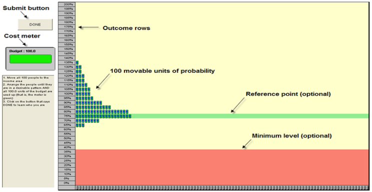

Figure: An implementation of the Distribution Builder

Application 2

Evaluating Hedge Fund Managers' Performance

- This can also be used to evaluate the performance of hedge fund managers (Amin & Kat)
- Assumption: Fund Managers try to produce a (time invariant) distribution, it can be estimated from the empirical distribution of the returns
- We try to replicate the distribution by investment in a reference index and cash
- Use the price of the replicating portfolio as measure of the fund's performance

Application 3

Rationalizing Investor's choice

- Investor invests according to its (not necessarily rational, e.g., behavioral) preferences V, with optimal portfoloio X^*
- Can we find a utility function U such that X^* is the optimal portfolio for expected utility maximization under U?
- We have by cost efficiency and duality

$$F^{-1}\big(1 - F_\xi(\xi)\big) = X^* = (U')^{-1}(\lambda\xi)$$

- Thus we have $U(x) = \int_c^x F_\xi^{-1}\big(1 - F(y)\big)\,dy$ (Bernard, Chen, Vanduffel)

Incomplete market

- One might ask how dependent these results are on the assumption of a complete market
- To do so, one has to establish a cost efficiency principle for incomplete markets
- The naïve guess would be that the optimizer X^* is a solution to

$$\inf_{Y \sim F} \sup_{\xi \in \Xi} \mathbb{E}\big[\xi Y\big] = \inf_{Y \sim F} c(Y)$$

  where the cost is given by the superhedging price $c(Y) = \sup_{\xi \in \Xi} \mathbb{E}\big[\xi Y\big]$

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- This is not correct

Incomplete market

Theorem

Assume that X^* is an optimizer of $\sup_{X \in \mathcal{X}} V(X_T)$ with cumulative distribution function F. Under some assumptions, X^* is also the almost surely unique solution to the following minimization problem

$$\inf_{Z \preceq_{cx} X^*} c(Z). \tag{1}$$

where $Z \preceq_{cx} X^*$ means that Z is smaller than X^* in convex order, i.e., for all convex functions v $\mathbb{E}[v(Z)] \leqslant \mathbb{E}[v(X^*)]$.

Assumption

- Standing Assumptions (for this talk)
  - The set Ξ of pricing kernels ξ is uniformly absolutely continuous
  - The support of V is bounded from below

Incomplete market

Idea of the Proof:

- We have to incorporate **superhedging**, over all pricing kernels $\xi \in \Xi$
- But how?

$$\inf_{Y \sim F} \sup_{\xi \in \Xi} \mathbb{E}\big[\xi Y\big]$$

or

$$\sup_{\xi \in \Xi} \inf_{Y \sim F} \mathbb{E}\big[\xi Y\big]$$

or are they even equivalent?

Incomplete Market

It turns out, the correct answer is

$$\sup_{\xi \in \Xi} \inf_{Y \sim F} \mathbb{E}[\xi Y] \tag{2}$$

Proposition

Assume that that the superhedging cost of some $Y \sim F$ is finite. Then (2) has a unique solution (ξ^*, Y^*) and Y^* can be expressed as

$$Y^* = \left(F^{-1} \circ (1 - F_{\xi*})\right)(\xi^*)$$

and

$$Y^* = \arg\max_{X \in \mathcal{X}} V(X_T)$$

Incomplete Markets

- How about

$$\inf_{Y\sim F} \sup_{\xi\in\Xi} \mathbb{E}[\xi Y]?$$

- Does a minimax principle hold?

$$\sup_{\xi\in\Xi} \inf_{Y\sim F} \mathbb{E}[\xi Y] \stackrel{?}{=} \inf_{Y\sim F} \sup_{\xi\in\Xi} \mathbb{E}[\xi Y]$$

this set is not convex

Convexification

Have to **convexify**: Denote the convex closure (wrt the topology of convergence in probability) of F-distributed random variables by

$$\overline{\mathrm{conv}}(F) := \overline{\mathrm{conv}\big(\{X \sim F\}\big)}^{L^0}$$

Proposition

Assume that F is integrable, then

$$\sup_{\xi \in \Xi} \inf_{Y \sim F} \mathbb{E}\big[\xi Y\big] = \sup_{\xi \in \Xi} \inf_{Y \in \overline{\mathrm{conv}}(F)} \mathbb{E}\big[\xi Y\big]$$
$$= \inf_{Y \in \overline{\mathrm{conv}}(F)} \sup_{\xi \in \Xi} \mathbb{E}\big[\xi Y\big] \leqslant \inf_{Y \sim F} \sup_{\xi \in \Xi} \mathbb{E}\big[\xi Y\big]$$

Convexification

Proposition

Assume that the probability space is a standard Borel space. Let $X \in L^0(\Omega, \mathcal{F}, \mathbb{P})$ and F be its cdf and assume that X is integrable. Without additional assumptions on the probability space,

$$\overline{\mathrm{conv}}(F) \subset \left\{ Y \in \mathcal{X} \,:\, Y \preceq_{cx} X \right\}$$

Furthermore if the probability space is atomless then

$$\overline{\mathrm{conv}}(F) = \left\{ Y \in \mathcal{X} \,:\, Y \preceq_{cx} X \right\}$$

Example

(Counter-)example for minimax without convexification:

Mixture SV model:

$$S_t = s_0 e^{\sigma W_t + (\mu - \frac{1}{2}\sigma^2)t}, \quad \sigma = \begin{cases} \sigma_H & \text{prob. } p \\ \sigma_L & \text{prob. } 1-p \end{cases}$$

Pricing kernels:

$$\xi_T^q = \frac{q}{p}\mathcal{E}\left(-\int_0^\cdot \frac{\mu - r}{\sigma_H} dW_t\right)_T \mathbb{1}_{\{\sigma=\sigma_H\}} + \frac{1-q}{1-p}\mathcal{E}\left(-\int_0^\cdot \frac{\mu - r}{\sigma_L} dW_t\right)_T \mathbb{1}_{\{\sigma=\sigma_L\}}$$

Choose $F \sim S_T$