Conic Asset Pricing and the Value of an Invested Dollar

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Joint Work with Wim Schoutens
The objective is to propose, investigate and report on the value of an invested dollar as an investment objective.

If the expected cash flow on a dollar equals the required return then the fair value of the dollar is unity.

The value may be above or below unity depending on whether the expected cash flow on the dollar is above or below the required return.

Hence, two entities determine the value of the invested dollar: (i) the expected cash flow on the dollar, and (ii) the required return on the dollar.
A widely used investment objective is the generation of alpha or the excess of the expected cash flow over the required return. However, from the perspective of the value of invested dollars pure alpha seeking strategies can be misdirected. This is because the value of the invested dollar only rises on increasing alpha per unit of required return. Required returns are traditionally modeled as proportional to a single beta. The value of the invested dollar is thus alpha per unit beta or the alpha beta ratio. Performance rankings based on these alpha beta ratios can be different from those based just on alpha.
Conic Required Returns

- Prices in financial markets must change to support the price with a return.
- The risk in markets requires an assessment on how far up and down they may fluctuate.
- These are addressed in abstract two price economies that model upper and lower prudent valuations.
- We introduce two risk characteristics from the theory of two price economies.
  - The first is the upper and lower valuation spread that we term a capital charge. These are the funds required to trade adversely in both directions.
  - The second is the depth of the lower valuation as an additional measure of downside risk.
- Required returns are modeled as compensating for capital charges and downside risk.
We find that in the presence of conic risk characteristics there is not much room or reason for covariation based risk assessments.

The asset beta is often insignificant, and when marginally significant it has the wrong sign.

Firm characteristics have been used along with beta’s in asset pricing tests, but they are available just monthly.

The risk characteristics may be constructed daily.

Furthermore, the risk characteristics have the same units as the expected cash flow to a dollar.
Black (1993) argued that estimating expected returns was far more important than explaining them.

He pointed out that estimation by averaging will often require very long data records and suggested that theory could help circumvent data deficiencies.

This becomes all the more important when the attention shifts to assessing tail risk on both sides of the risk spectrum.

M. (2017) shows how using knowledge about the full uncentered distribution improves the efficiency of expected return estimation beyond averaging.

We therefore make our computations using both the raw data and models fit to the raw data.
Modeling the Uncentered Return Distribution

- We have noted that prices must change to provide a return.
- Furthermore, the change must be a surprise and hence occur at surprise times.
- The process of price changes is thus pure jump.
- Given the large number of shocks anticipated we also take it to be limit law.
- All the pure jump limit laws are infinitely divisible with a special structure for their jump arrival rate functions.
- The arrival rates scaled by the absolute jump size are decreasing functions of the absolute jump size.
- They constitute the self decomposable random variables. (Lévy (1937), Khintchine (1938)).
Distributions come into existence at a horizon, be it a minute, day, week, month or year.

At longer horizons we lose data.

At any horizon distributions are complicated if and when they are available.

At shorter horizons probabilities of tail events are converging to zero, but their ratio to the horizon can have a well defined limit.

This limit is the arrival rate function that can be a lot simpler than the densities involved.
Arrival Rate Functions

- For limit laws the aggregate arrival rate is infinite and the process has infinitely many jumps in any interval.
- For pure jump self decomposable laws the risks may be represented by their arrival rate functions at the instantaneous level.
- We then get away from densities, distributions and moments.
- They are replaced by integrals of jump responses with respect to arrival rates that are called variations.
Results at three levels

- We present results at three levels, the raw data at the daily level.
- The use of probability models for the daily return.
- The use of limiting arrival rate functions.
- The best and most stable results turn out to be at the level of arrival rate functions.
- I was going to just work at the level of arrival rate functions but my coauthor persuaded me to include the other two.
- Now we can see the differences. (Thanks Wim).
- My bigger agenda in risk analysis is to forget random variables and their densities and always represent risk by arrival rate functions.

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The Probability Model

- A particularly simple, tractable and empirically adequate self-decomposable pure jump law is the variance gamma model.
- The jump size scaled arrival rate function is just a negative exponential and clearly decreasing.
- Let $X$ be the variance gamma random variable then the characteristic function of $X$ is (M. Carr and Chang (1998))

$$E \left[ \exp \left( iuX \right) \right] = \left( \frac{1}{1 - iu\theta \nu + \sigma^2 \nu u^2 / 2} \right)^{1/\nu}.$$

- The random variable $X$ can be thought as generated according to

$$X = \theta U + \sigma \sqrt{U} Z,$$

where $U$ is gamma distributed with shape and scale parameters equal to $1/\nu$, $\nu$ and $Z$ is a standard normal variate.
Carr, Geman, M. and Yor (2002) developed the arrival rate function for the variance gamma in the form

\[ k(z) = C \frac{\exp(-Mz)}{z}; \quad k(-z) = C \frac{\exp(-Gz)}{z}, \quad z > 0. \]

It is clear that the variance gamma random variable is a self decomposable random variable with a simple negative exponential structure for the functions \( zk(z), zk(-z) \) for \( z > 0 \).

The mapping from the \( C, G \) and \( M \) parameters to \( \sigma, \nu \) and \( \theta \) is

\[ \nu = \frac{1}{C}, \quad \theta = C \left( \frac{1}{M} - \frac{1}{G} \right), \quad \sigma = \sqrt{\frac{2C}{GM}}. \]
VG Density

- The density has a closed form (Carr and M. (2014)) given by

  \[ f_{CGM}(x) = \frac{(GM)^C}{2^{C-1} \Gamma(C) \sqrt{2\pi} \left( \frac{G+M}{2} \right)^{C-1/2}} \exp \left( \frac{G - M}{2} x \right) |x|^{C-1/2} \]

  \[ \times K_{C-1/2} \left( \frac{G + M}{2} |x| \right) \]

  where \( K_{\nu}(x) \) is the modified Bessel function.

- If \( R \) is the rate of return per unit time then

  \[ E[R] = \left( \frac{1}{1 - \theta \nu - \sigma^2 \nu} \right)^{1/\nu} - 1 = \left( \frac{GM}{(G+1)(M-1)} \right)^C - 1. \]
Prudential Valuation

- Given the log price relative for a horizon $h$ at the level $x$ with return density $f(x, h)$ the expected cash flow to a dollar is
  \[ R = \int_{-\infty}^{\infty} (e^x - 1) f(x, h) \, dx. \]

- Making the change of variable to $a = (e^x - 1)$ and performing an integration by parts in $a$ we obtain that
  \[ R = -\int_{-\infty}^{0} da \int_{e^x-1<a} f(x, h) \, dx + \int_{0}^{\infty} da \int_{e^x-1>a} f(x, h) \, dx. \]

- Denoting the physical probabilities by $P$ and defining
  \[ P(A) = \int_{A} f(x, h) \, dx. \]

- In terms of $P$ we may write
  \[ R = -\int_{0}^{\infty} P\left((e^x - 1)^- > a\right) \, da + \int_{0}^{\infty} P\left((e^x - 1)^+ > a\right) \, da. \]

- Expectation is an integral of tail probabilities.
Expectation with respect to nonadditive probability

- Expectation treats all probabilities equally
- Prudence suggests that probabilities of extreme tail events should be taken with a grain of salt.
- Lower prudential valuations attain conservatism by employing a concave distribution function \( \Psi(u) \geq u, 0 \leq u \leq 1 \).
- The distortions lift the probabilities of the loss events on the left while simultaneously employing the convex distortion

\[
\hat{\Psi}(u) = 1 - \Psi(1 - u) \leq u,
\]

- to lower the gain probabilities on the right.
The lower prudential valuation based on probability distortions is then

\[ L = - \int_0^\infty \Psi \left( P \left( (e^x - 1)^- > a \right) \right) da + \int_0^\infty \hat{\Psi} \left( P \left( (e^x - 1)^+ > a \right) \right) da \]

The upper prudential valuation applied to modeling liability add-ons operate the other way, lifting the right tail and lowering the left tail to obtain

\[ U = - \int_0^\infty \hat{\Psi} \left( P \left( (e^x - 1)^- > a \right) \right) da + \int_0^\infty \Psi \left( P \left( (e^x - 1)^+ > a \right) \right) da \]
Selection of Parametric Distortion

- The greater the concavity of $\Psi$ or the convexity of $\hat{\Psi}$, the smaller is the lower valuation and the larger is the higher valuation.

- A single parameter distortion termed $\text{minmaxvar}$ was introduced in Cherny and M. (2009) and is given by

$$\Psi^{(\gamma)}(u) = 1 - \left(1 - u^{\frac{1}{1+\gamma}}\right)^{1+\gamma},$$

where the parameter $\gamma$ controls the concavity or stress level of the distortion.
The lower and upper valuations may be seen to be infima and suprema of expectations under alternative test measures $J \in \mathcal{M}$.

In this case it may be observed that

$$L = \inf_{J \in \mathcal{M}} E^J [(e^x - 1)]$$

$$U = \sup_{J \in \mathcal{M}} E^J [(e^x - 1)].$$

The lower and upper valuations are then also based on infima and suprema of numerous covariations for it is also the case that

$$L = \inf_{J \in \mathcal{M}} E \left[ \frac{dJ}{dP} (e^x - 1) \right]$$

$$U = \sup_{J \in \mathcal{M}} E \left[ \frac{dJ}{dP} (e^x - 1) \right].$$
The set of test measures $\mathcal{M}$ in this case given by all probabilities $J$ with the property

$$J(A) \leq \Psi(P(A)), \text{ all } A.$$ 

Hence the greater the stress level or concavity of $\Psi$ the wider the class of test measures and the smaller or larger are the lower or the upper valuations.
The variance gamma model was fit to the S&P 500 return data for 252 days prior to September 28, 2015.

The estimated parameter values for $\sigma$, $\nu$ and $\theta$ were respectively 0.0095, 0.8552 and 7.947 basis points.

We present a graph of the tail probabilities observed in the data.

The fitted model tail probabilities.

The distorted tail probabilities for the lower and upper prudential valuation for the stress level of 0.15 using $\text{minmaxvar}$.

The three values, lower, expectation and upper are respectively in basis points $-33.15$, $-7.50$ and $16.78$. 
Expectations and their Distortions

Tail Probabilities

Cash Flow to a Dollar

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Conic Asset Pricing

70th Birthday Conference Department of Finance and Risk Engineering New York University Tandon School of Engineering Joint Work with Wim Schoutens
Suppose that the price process $S(t)$ has a logarithm $X(t)$ that is a pure jump process.

Let $m(dx, dt)$ be the random measure for the jumps in $X(t)$.

The random measure $m$ counts all the jumps that occur and under finite variation for $X(t)$ it is the case that

$$X(t) = \int_0^t \int_{-\infty}^{\infty} x m(dx, ds)$$

$$= \sum_{s \leq t} \Delta X(s).$$
Further suppose the random measure $m$ has a compensator that is a deterministic function of the jump size and time given by 

$$\tilde{k}(x, s)dxds.$$ 

By construction 

$$S(t) = S(0) \exp(X(t))$$ 

The continuously compounded return to time $h$, defined by $\mu(h)$ is 

$$S(0) \exp(\mu(h)) = E[S(h)].$$ 

It follows that 

$$\exp(\mu(h)) = E[\exp(X(h))].$$
Exponential Variations

- From the expression for the characteristic function we deduce that

\[ E \left[ \exp \left( X(t) \right) \right] = \exp \left( \int_{0}^{t} \int_{-\infty}^{\infty} (e^{x} - 1) k(x, s) \, dx \, ds \right) \]

- and equivalently

\[ \mu(h) = \int_{0}^{h} \int_{-\infty}^{\infty} (e^{x} - 1) k(x, s) \, dx \, ds. \]

- Hence it is the case that the exponential variation is

\[ \int_{-\infty}^{\infty} (e^{x} - 1) k(x, 0) \, dx = \lim_{h \downarrow 0} \frac{\mu(h)}{h}. \]

- This motivates the use of the exponential variation as the continuous time concept for reward measured strictly in terms of the arrival rate function.
The reward in the continuous limit is the exponential variation of the cash flow to a dollar,

\[ V = \int_{-\infty}^{\infty} (e^x - 1) k(x) \, dx. \]

Applying integration by parts one may write the variation as the integral of tail measures.

More specifically on making the change of variable \( a = (e^x - 1) \) and performing an integration by parts in \( a \) one shows that

\[
V = - \int_{-\infty}^{0} \int_{(e^x - 1) < a} k(x) \, dx \, da + \int_{0}^{\infty} \int_{(e^x - 1) > a} k(x) \, dx \, da \\
= - \int_{0}^{\infty} K \left( (e^x - 1)^- > a \right) \, da + \int_{0}^{\infty} K \left( (e^x - 1)^+ > a \right) \, da
\]
Measure Distortions

- Introduce two increasing functions $\Lambda^+$ and $\Lambda^-$ above and below the identity respectively, with the former concave and the latter convex to define the lower variation by

$$V_L = -\int_0^1 \Lambda^+ \left( K \left( (e^x - 1)^- > a \right) \right) da$$

$$+ \int_0^\infty \Lambda^- \left( K \left( (e^x - 1)^+ > a \right) \right) da.$$  

- For the prudent upper variation the lift occurs on the right and the reduction on the left to get

$$V_U = -\int_0^1 \Lambda^- \left( K \left( (e^x - 1)^- > a \right) \right) da$$

$$+ \int_0^\infty \Lambda^+ \left( K \left( (e^x - 1)^+ > a \right) \right) da.$$
One may define two increasing concave functions $\Gamma^+, \Gamma^-$ and define

$$
\Lambda^+(x) = x + \Gamma^+(x) \\
\Lambda^-(x) = x - \Gamma^-(x).
$$

Note that $\Lambda^-(x) \leq x$ requires $\Gamma^-(x) \geq 0$.

Parametric forms for the functions $\Gamma^+, \Gamma^-$ were employed in Eberlein, M. Pistorius and Yor (2014) with

$$
\Gamma^+(x) = a \left(1 - e^{-cx}\right) \frac{1}{1+\gamma} \\
\Gamma^-(x) = \frac{b}{c} (1 - e^{-cx}).
$$

The lower and upper prudent variations can be expressed as infima and suprema over measures $L$ satisfying for all $A \subset \mathbb{R} - \{0\}$ that

$$
L(A) \in \left[\Lambda^-(K(A)), \Lambda^+(K(A))\right].
$$
In the VG case the inner integrals may be done analytical to obtain

\[ V_L = - \int_0^1 \Lambda^+ \left( Cexpint \left( -G \log \left( 1 - u \right) \right) \right) du + \int_0^\infty \Lambda^+ \left( Cexpint \left( M \log \left( 1 + u \right) \right) \right) du \]

A similar analysis shows that the upper prudential valuation is

\[ V_U = - \int_0^1 \Lambda^- \left( Cexpint \left( -G \log \left( w \right) \right) \right) dw + \int_1^\infty \Lambda^+ \left( Cexpint \left( M \log \left( w \right) \right) \right) dw \]
For the S&P 500 index for September 28, 2015 reported on in section 5 the \( C, G \) and \( M \) parametrization is respectively 1.1693, 170.0209 and 152.4098.

For the measure distortion parameters \( a, b, c \) and \( \gamma \) respectively at 3.1191, 0.9694, 0.3108 and 0.4877 we show the tails measures and the distorted tail measures whose integrals appropriately differed define the variation and the prudent lower and upper variations.

The lower and upper prudent variations were in basis points \(-153.43\) and \(176.71\) while the exponential variation was 8.40.
Figure: Tail Measures and their distortions. The tail measures are shown in blue. The lower distortion is in black and the upper distortion is in magenta.
Market concerns about how far up and down prices may go vary over time.

The use of fixed or frozen distortions can lead to unrealistic levels of required returns that get reflected in the value of an invested dollar drifting away from unity.

These considerations and observations led us to develop methods for calibrating distortion parameters to market data.

The calibration principle adopted here is to distort sufficiently to allow the risk neutral measure on S&P 500 index options to be among the test measures for the distorted physical measure on the S&P 500 index.
Calibrating Distortions

- Denote by $F_N(x)$ this risk neutral distribution function as a function of $x = \ln(S(t + H)/S(t))$ for some horizon $H$ in days close to one month.
- Let $F_P(x)$ be the corresponding physical distribution function.
- The condition for the risk neutral probability to be among the test probabilities is that
  \[ F_N(x) \leq \Psi(\gamma)(F_P(x, H)) \]
- Equivalently the complementary probability $G_N(x)$ must satisfy
  \[ G_N(x) = 1 - F_N(x) \]
  \[ \geq 1 - \Psi(\gamma)(F_P(x, H)) \]
  \[ = 1 - \Psi(\gamma)(1 - G_P(x, H)) \]
  \[ = \Psi(\gamma)(G_P(x, H)) \]
Let \( X_h \) be the self decomposable random variable at the short daily horizon.

By self decomposability for every constant \( \kappa, 0 < \kappa < 1 \) there exists an independent random variable \( X_h^{(\kappa)} \) such that

\[
X_h^{(d)} = \kappa X_h + X_h^{(\kappa)}.
\]

The long horizon return suggested in Eberlein and M. (2010) is to define \( X_H \) as the sum of running \( (\kappa X_h) \) as a Lévy process to time \( H \) and scaling the independent component \( X_h^{(\kappa)} \).

Hence we propose to take

\[
X_H^{(d)} = (\kappa X_h)_{H-h} + \left( \frac{H}{h} \right)^\eta X_h^{(\kappa)}.
\]
One may simulate from $X_H$ directly as the law of $X_h^{(κ)}$ may be identified and simulated from.

The random variable $X_H$ is not in the variance gamma class but we approximate it by a variance gamma law by fitting a variance gamma model to the simulated data.

This procedure gives us $F_P(ξ, H)$ in the variance gamma class.

We follow Eberlein and M. (2010) and employ the suggested values for $κ = 0.5$ and $η = 0.5$. 
Calibration of Measure Distortion

- By estimating the risk neutral distribution at the monthly maturity in the variance gamma class we obtain a risk neutral arrival rate function $k_N(x)$.
- A variance gamma approximation to the long horizon return delivers a physical arrival function $k_P(x)$.
- For the calibration of measure distortions the conditions mirror those for probability distortions and require that for all $a > 0$

\[
\int_{-\infty}^{-a} k_N(x) \, dx \leq \Gamma^+ \left( \int_{-\infty}^{-a} k_P(x) \, dx \right)
\]
\[
\int_{a}^{\infty} k_N(x) \, dx \geq \Gamma^- \left( \int_{a}^{\infty} k_P(x) \, dx \right).
\]
Sample Calibration of Probability Distortion

- For the S&P 500 index on October 13, 2016 using 252 days of prior continuously compounded daily return data we estimate by matching digital moments the variance gamma model.
- We present the observed and fitted tail probabilities.
Digital Moment Estimation of VG on SPX 20161013

sg 0.0093
nu 0.9910
th 0.0008
The Distribution Function at Month End

- The number of days to the option maturity in this case was 30 days.
- The long horizon 30 day returns were generated by shaving the daily variance gamma law by a half running this as an i.i.d. process for 30 days and adding to this the independent component scale by $\sqrt{30}$.
- The variance gamma approximation to the long horizon return had the parameters

$$\sigma = 0.0493; \quad \nu = 0.6598; \quad \theta = 0.0054.$$
Lifting the Left and Dropping the Right

The corresponding risk neutral probabilities were obtained directly from put and call prices by evaluating the derivatives of call and put prices.

Distorted down probabilities were constructed to attain domination from above of the risk neutral probabilities.

Distorted up probabilities were constructed to attain domination from below of risk neutral probabilities.

The two stress levels were 0.3360 and 0.3847.

The stress level for the day is the larger of the two.
Calibration of Probability Distortion

down stress 0.3360
up stress 0.3847

sg 0.0093 nu 0.9910 th 0.0008
sgH 0.0493 nuH 0.6598 thH 0.0054
For the option data on October 13, 2016 on the S&P 500 index we fit the VGSSD model for two maturities close to one and three months. We present the fit of the model to observed option prices along with the Sato process parameters.
Risk Neutral Calibration of VGSSD to SPX 20161013

sg 0.1293
nu 1.4721
th -0.1147
rho 0.5586
Measure Distortion Parameters

- From this fit one may identify a risk neutral tail measure in the variance gamma class for the one month maturity.
- The corresponding physical tail measure in the variance class was already identified from the long horizon approximation introduced earlier.
- For the distortions $\Lambda^+, \Lambda^-$ we need to identify four parameters, $a, b, c, \gamma$.
- By symmetry we set $a = b / c$. We define $b$ by recognizing that $\Lambda^{-'}(0) = 1 - b$.
- Equating $\Lambda^{-'}(0)$ to the derivative of the risk neutral measure at zero identifies $b$.
- The parameter $c$ is selected to accomplish the required domination on the right while the parameter $\gamma$ is selected to attain the domination on the left.
Calibration of Measure Distortions

- Down Tail Measure P
- Up Tail Measure P
- Down Tail Measure Q
- Up Tail Measure Q
- Down Distorted Measure
- Up Distorted Measure

\[ a = 4.0932 \]
\[ b = 0.9837 \]
\[ c = 0.2403 \]
\[ g = 0.7279 \]
The probability and measure distortions were calibrated to data on S&P 500 index options from January 4, 2010 to March 3, 2016. For each of 1552 days in this time period for all of 229 stocks, 9 sector ETF’s and the S&P 500 index or a total of 239 underlying assets the lower and upper prudential valuations were computed. The computations were based on the empirical tail probabilities, the fitted variance gamma tail probabilities and the measure distorted fitted variance gamma arrival rate functions. In each of these three cases two risk characteristics were constructed. The first is a spread or capital charge measure defined as

\[ s_{it} = \ln \left( \frac{1 + U_{it}}{1 + L_{it}} \right). \]

The second is downside risk measure defined as

\[ d_{it} = -\ln(1 + L_{it}). \]
A cross sectional regression was run each day across the 239 underlying assets where the dependent variable was the mean daily percentage change over the last 252 days for the case of the raw data with no probability model.

For the variance gamma model fitted by matching tail probabilities the dependent variable was the expected return.

For the measure distortions the dependent variable was the exponential variation.

The independent variables were in addition to a constant, the spread and downside risk characteristics.

These are computed in the three cases by distortion of the empirical tail probabilities, the model tail probabilities, and finally measure distortion of the arrival rate functions.
We present the three sets of $RSQ'$s for the 1552 days.

We observe that the best performance is offered by the use of measure distortions.

The distortion of empirical tail probabilities gives a better result than the distortion of model tail probabilities.
We present a graph of the coefficient for the capital risk characteristic using the three regressions.

The results from the use of measure distortions are the most stable and close to a half for a fair while.

The results from the probability distortions are quite volatile and the empirical tail probability distortions are also volatile but less so.
We present the coefficients for the characteristic on downside risk exposure.

The probability distortion coefficient is quite volatile and is presented separately.

The empirical tail probability often has a negative coefficient possibly due to the limited reach of data sets.

The measure distortion is quite stable around the 2% mark.
The results for the $R^2$ were similar to those without the addition of the asset beta.

The capital and down risk premia were also similar.

Table 2 presents the quartiles for the t-statistics on the three coefficients.

With the raw data beta is generally positive but not significant.

With the probability distortion and measure distortion when significant it has the wrong sign.
<table>
<thead>
<tr>
<th></th>
<th>quartiles</th>
<th></th>
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</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>Raw Capital</td>
<td>18.74</td>
<td>33.22</td>
<td>242.3</td>
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<tr>
<td>Data DownRisk</td>
<td>-3.70</td>
<td>-2.15</td>
<td>-0.93</td>
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</tr>
<tr>
<td>Beta</td>
<td>0.42</td>
<td>1.65</td>
<td>4.33</td>
<td></td>
</tr>
<tr>
<td>Probability Capital</td>
<td>2.84</td>
<td>5.35</td>
<td>8.12</td>
<td></td>
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<tr>
<td>Model DownRisk</td>
<td>4.68</td>
<td>6.23</td>
<td>7.73</td>
<td></td>
</tr>
<tr>
<td>Beta</td>
<td>-3.12</td>
<td>-1.83</td>
<td>0.52</td>
<td></td>
</tr>
<tr>
<td>Measure Capital</td>
<td>100.4</td>
<td>207.2</td>
<td>352.6</td>
<td></td>
</tr>
<tr>
<td>Distortion DownRisk</td>
<td>14.39</td>
<td>26.26</td>
<td>36.97</td>
<td></td>
</tr>
<tr>
<td>Beta</td>
<td>-4.11</td>
<td>-2.62</td>
<td>-2.14</td>
<td></td>
</tr>
</tbody>
</table>
Relative Contribution of Capital Charges

- For 1552 * 239 asset days we evaluated the share of the positive required return made up by the contribution of the capital charge.
- The proportion of positive required returns was 94.34, 45.19 and 84.90 percent for measure distortion, probability distortion, and the raw data.
- For the measure distortion the quartiles were 67.13, 78.39 and 86.01 respectively.
- The corresponding values for the probability distortion applied to the probability model were 7.48, 18.19 and 43.09.
- The use of the empirical distribution had proportions of 85.40 and 100 percent.
- We conclude that the use of the empirical distribution is curtailed in its ability to address downside risk.
- The probability distortion has considerable weight on downside risk probably as a consequence of a strong distortion.
With regard to the value of invested dollar we present the probability distribution of this value across the 370,928 asset days using the raw data, the probability model and the measure distorted arrival rate functions.

The most stable results are with the measure distorted arrival rate functions followed by the probability model and the then the raw data that has a long negative tail which was truncated for presentation.
Value of Invested Dollar Probability Model

Probability vs. Value of Invested Dollar

Probability $\times 10^{-3}$

Value of Invested Dollar
Each day the assets may be ranked on the basis of alpha or the value of invested dollar as evaluated by the alpha beta ratio, the measure distortion, probability distortion or the distortion of the empirical distribution function.

These rankings may be termed, $a$, $ab$, $m$, $p$ and $e$ for the five ways in which they may be generated.

Table 3 presents the quartiles across the 1552 days for the correlations between these five rankings.

Apart from alpha and the alpha beta ratio the value of the invested dollar evaluated via measure distortions, probability distortions or the empirical distribution deliver different rankings.
**TABLE 3**

Ranking Correlations

<table>
<thead>
<tr>
<th>Quartiles</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>a,ab</td>
<td>0.9770</td>
<td>0.9822</td>
<td>0.9877</td>
</tr>
<tr>
<td>a,m</td>
<td>0.1175</td>
<td>0.1917</td>
<td>0.2829</td>
</tr>
<tr>
<td>a,p</td>
<td>0.3352</td>
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Risk characteristics based on prudent upper and lower valuations as they appear in abstract models of two price economies are introduced as explanatory variables to be compensated for in forming required returns.

They are then used to formulate required returns.

The value of the invested dollar is modeled as the ratio of the expected return to the required return.

The spread and downside market depth is modeled using parametric models for probability distortions at the level of daily data.

In the continuous time limit one applies parametric measure distortions to the tail measures of jump arrival rates.

Both the probability and measure distortions are calibrated to the market for options on the S&P 500 index.
Conclusion II

- It is observed that frozen distortion parameters can lead to drifts in the value of the invested dollar away from unity probably due to obsolete or unrealistic distortion stress levels.
- The results show a better performance delivered on employing the analytical structure of measure distorting jump arrival rates.
- The two risk characteristics are both highly significant in explaining expected returns especially when applying measure distortions.
- There is also little room or reason for the inclusion of betas or possibly other covariation based measures once risk characteristics have been introduced.
- The relative contribution of capital charges to required returns has an interquartile range of 67 to 86 percent when assessed by measure distortions applied to arrival rate functions.
- The different constructions for the value of an invested dollar also deliver different asset rankings across time.