Vol, Skew, & Smile Trading

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Some Rough Definitions

- **Implied Volatility** (IV) is roughly defined as the unique volatility input fed to the Black Merton Scholes (BMS) option pricing formula which is consistent with an observed market option price.
- Loosely speaking, a *vol trade* is any option position that gains on average if the realized volatility or variance of the underlying is sufficiently high relative to an option’s implied volatility or variance.
- Practitioners roughly define *IV skew* as some measure of the slope of an implied vol or variance curve that arises when co-terminal IV’s are graphed against some measure of the options’ moneyness.
- Loosely speaking, a *skew trade* is any option position that gains on average when the realized covariation of implied vols with the underlying’s return is sufficiently high relative to the skew.
- Practitioners also roughly define *IV smile* as some measure of the convexity of an implied vol or variance curve that arises when co-terminal IV’s are graphed against some measure of moneyness.
- Loosely speaking, a *smile trade* is an option position that gains on average when “vol of vol” is sufficiently high relative to the smile.
Some Motivating Questions

- Which options should you trade so as to be profitable on average when either:
  1. you know realized vol will definitely exceed 10% and yet ATM implied vol is currently below 10%, or
  2. you know that the correlation of every implied vol with the underlying will realize positive and yet the OTM call implied vol is currently below an equally OTM put implied vol, or
  3. you know that implied vol’s are themselves volatile and yet three implied vol’s currently plot linearly?

- We actually respectively consider vol, skew, or smile trading under a view on the realized variance of the log of the underlying, the realized covariation of the log underlying with log implied vol, or the realized variance of log implied vol. Given a view on one, we assume no clue on the other two.
Financial Setting

- Working in a foreign exchange (FX) context, we assume zero interest rates throughout this talk.
- Let $S_t > 0$ be the underlying spot FX rate at time $t \in [0, T]$ expressed as domestic currency per foreign currency unit.
- Let $I_t(K)$ denote the Black Merton Scholes implied vol at varying strike rate $K > 0$ for some fixed maturity date $T \geq t \geq 0$.
- We suppose that a market maker continuously quotes the FX level $S_t$ and an entire implied vol curve $I_t(K), K > 0$.
- We will be assuming that the risk-neutral dynamics of the FX rate and all co-terminal implied vol’s are driftless geometric Brownian motions, generalized to have arbitrary unknown stochastic volatility.
- We assume no frictions, in particular markets are always open and bid ask spreads vanish.
Implications of No Arbitrage

- We assume no arbitrage between the two currencies, but we allow a possible arbitrage whenever an option is involved.
- The absence of arbitrage between the two currencies implies the existence of two equivalent martingale measures $\mathbb{Q}^-$ and $\mathbb{Q}^+$. 
- Under $\mathbb{Q}^-$, $S$ is a positive (non-trivial local) martingale, while under $\mathbb{Q}^+$, $1/S$ is a positive (non-trivial local) martingale.
- Since $S$ is risky, $\ln S$ has negative drift under $\mathbb{Q}^-$, but positive drift under $\mathbb{Q}^+$. 

The Black Merton Scholes Model

- Recall that under $\mathbb{Q}_-$, $S > 0$ is a (non-trivial local) martingale.
- The Black Merton Scholes (BMS) model further restricts $S$ to have continuous paths and to have constant volatility.
- To see what this means, let $\mathbb{Q}_b^\pm$ be the two equivalent martingale measures $\mathbb{Q}_-$ and $\mathbb{Q}_+$ in the BMS model.
- Recall we assume zero interest rates over $[0, T]$.
- Under $\mathbb{Q}_b^-$, the BMS model assumes that $S$ solves the following stochastic differential equation (SDE):

$$dS_t = \sigma S_t dW_t, \quad t \in [0, T],$$

where $\sigma > 0$ is the constant instantaneous volatility of $S$. Here, $W$ is a $\mathbb{Q}_-$ standard Brownian motion.
Let the time to maturity $\tau \equiv T - t$ be the difference between the fixed maturity date $T > 0$ and the moving calendar time $t \in [0, T]$. 

For $\sigma > 0$, $\tau > 0$, and 0 int. rates, the BMS put pricing formula is:

$$
P^b(S, \sigma, \tau; K) \equiv KN(z_-(K/S, \sigma \sqrt{\tau})) - SN(z_+(K/S, \sigma \sqrt{\tau})),$$

where for $z \in \mathbb{R}$, $N(z) \equiv \int_{-\infty}^{z} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$ is the standard normal cumulative distribution function, whose arguments are defined as:

$$z_{\pm}(K/S, \sigma \sqrt{\tau}) \equiv \frac{\ell_{\pm}(K/S, \sigma \sqrt{\tau})}{\sigma \sqrt{\tau}}, \quad \ell_{\pm}(K/S, \sigma \sqrt{\tau}) \equiv \ln(K/S) \pm \sigma^2 \tau/2.$$

$\ell_{\pm}(K/S, \sigma \sqrt{\tau}) = \ln(K/S) - E_{Q_{\pm}}^b[\ln(S_T/S_t)|S_t = S]$ is called log moneyness.
BMS Implied Volatilities

- At all times \( t \in [0, T] \), our market-maker quotes BMS implied volatility (IV) by strike, \( I_t(K) \) for all strike rates \( K > 0 \) and for a fixed maturity date \( T > 0 \).

- Let \( P_t(K) \) be the time \( t \) market price of the put option with strike rate \( K > 0 \) and maturity date \( T \geq t \).

- By the definition of BMS IV, the time \( t \) market price of the put is:

\[
P_t(K) = P^b(S_t, I_t(K), T - t; K), \quad K > 0, t \in [0, T].
\]

- For each fixed strike \( K > 0 \), IV is the wrong volatility to put into the wrong put pricing formula \( P^b \) to get the right put price \( P_t(K) \).
Cash Gamma in the BMS Model

- Practitioners call partial derivatives of the option price “greeks”.
- For example, differentiating the BMS put option pricing formula twice w.r.t. $S$, the put’s gamma is a greek given by:

\[
P_{11}^b(S, \sigma, \tau; K) = \frac{N'(z_+ (K/S, \sigma \sqrt{\tau}))}{S \sigma \sqrt{\tau}} , \quad S > 0, \sigma > 0, \tau > 0, K > 0.
\]

- We define the cash gamma of the put as:

\[
S^2 P_{11}^b(S, \sigma, \tau; K) = S \frac{N'(z_+ (K/S, \sigma \sqrt{\tau}))}{\sigma \sqrt{\tau}} = K \frac{N'(z_- (K/S, \sigma \sqrt{\tau}))}{\sigma \sqrt{\tau}}.
\]

- Cash gamma is measured in the same units as the option premium, i.e. domestic currency. All of the cash greeks will have this important property.
Other Greeks in the BMS Model

- We will be interested in all the greeks that arise from an application of Itô’s formula in our setting.
- It turns out that all of the greeks that we are interested in have formulas that are just simple multiples of the formula for cash gamma on the last slide.
- Letting $\Gamma \equiv S^2 \mathcal{P}_{11}^b(S, \sigma, \tau) = \frac{KN'(z_-)}{\sigma \sqrt{\tau}}$ denote cash gamma, the table below lists the greeks that we are interested in:

<table>
<thead>
<tr>
<th>Name</th>
<th>Definition</th>
<th>Link to Cash Gamma $\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cash Vega</td>
<td>$\sigma \mathcal{P}_{2}^b(S, \sigma, \tau)$</td>
<td>$\Gamma \sigma^2 \tau$</td>
</tr>
<tr>
<td>Cash Vanna</td>
<td>$S \sigma \mathcal{P}_{12}^b(S, \sigma, \tau)$</td>
<td>$\Gamma \ell_-$</td>
</tr>
<tr>
<td>Cash Volga</td>
<td>$\sigma^2 \mathcal{P}_{22}^b(S, \sigma, \tau)$</td>
<td>$\Gamma \ell_- \ell_+$</td>
</tr>
<tr>
<td>Theta</td>
<td>$-\mathcal{P}_{3}^b(S, \sigma, \tau)$</td>
<td>$-\Gamma \frac{\sigma^2}{2}$</td>
</tr>
</tbody>
</table>

where

$$\ell_{\pm}(\frac{K}{S}, \sigma \sqrt{\tau}) \equiv \ln(\frac{K}{S}) \pm \sigma^2 \tau / 2$$

is a log moneyness measure.
Dynamical Restrictions on Spot FX Rate

• Until now, we have not imposed any dynamical restrictions. We have merely presented the zero-rates BMS-model European-put pricing formula, defined implied volatility, and calculated several greeks of interest.

• We now restrict the dynamics of the underlying spot FX rate $S$ and the implied volatility curve $I(K)$, $K > 0$. Suppose that under $Q$ and zero interest rates, $S$ solves the following SDE:

$$dS_t = \sigma_t S_t dW_t, \quad t \in [0, T],$$

where recall $W$ is a $Q$ standard Brownian motion.

• The subscript $t$ on $\sigma$ indicates that the instantaneous volatility of $S$ is now a stochastic process. We treat $\sigma_0$ as a positive random variable whose $Q$ law is unknown. We do not assume that investors are able to directly specify the $Q$ dynamics of the instantaneous volatility process $\sigma$. 
Dynamical Restrictions on Implied Volatilities

- We assume instead that the IV curve $I_t(K), K > 0$ solves:

  $$dl_t(K) = \omega_t I_t(K) dZ_t, \quad K > 0, t \in [0, T],$$

  where $Z$ is a one-dimensional $Q$-standard Brownian motion.

- We assume that the initial IV curve $I_0(K), K > 0$ is an observed strictly positive function and that the volvol process $\omega_t \in (0, \infty)$ is an unobserved positive bounded process.

- As a result, subsequent IV curves $I_t(K), K > 0, t \in (0, T]$ are all strictly positive.

- The spot FX rate $S$ and the IV curve $I_t(K), K > 0$ both follow driftless geometric Brownian motion under $Q$, generalized to have arbitrary unknown stochastic volatility.
Driftless Strike-Invariant Proportional Shifts

• Recall the assumed risk-neutral dynamics for the IV curve $I_t(K)$, $K > 0$:

\[ dl_t(K) = \omega_t I_t(K) dZ_t, \quad K > 0, t \in [0, T], \]

where $Z$ is a one-dimensional $\mathbb{Q}$-standard Brownian motion.

• Since the volvol process $\omega_t$ does not depend on $I$ or $K$, all of the implied volatilities undergo the same proportional shifts.

• We say that implied volatilities undergo driftless strike-invariant proportional shifts.

• Our dynamics imply that the ratio of two IV’s at any future time is just the ratio of the two initial IV’s:

\[ \frac{I_t(K)}{I_t(K')} = \frac{I_0(K)}{I_0(K')}, \quad K \neq K', t \in (0, T]. \]
Correlation and Covariation Processes

- Recall our dynamical restrictions on $S$ and $I(K)$, $K > 0$:
  \[ dS_t = \sigma_t S_t dW_t, \quad dl_t(K) = \omega_t l_t(K) dZ_t, \quad K > 0, \; t \in [0, T], \]
  where $W, Z$ are $\mathbb{Q}$-standard Brownian motions (SBM's).
- The volatilities of $S$ and $I(K)$ are themselves strictly positive stochastic processes, whose initial levels $\sigma_0$ and $\omega_0$ and subsequent dynamics $d\sigma_t$ and $d\omega_t$ are unspecified random variables.
- Let $\rho_t \in [-1, 1]$ be the bounded stochastic process governing the correlation between the 2 SBM's $W$ and $Z$ at time $t \in [0, T]$:
  \[ d\langle W, Z \rangle_t = \rho_t dt. \]
  We assume that $\rho_0$ is a random variable whose law has support in $[-1, 1]$, but whose $\mathbb{Q}$-law is unknown.
- The covariation process $\gamma_t$ is defined as the coefficient of $dt$ in $d\langle \ln S, \ln I(K) \rangle_t$. The SDE's imply $\gamma_t \equiv \sigma_t \rho_t \omega_t$, which are all unspecified.
At-the-Money Strike Rate, IV, and Straddle

• Recall our log moneyness measure \( \ell_-(\frac{K}{S}, \sigma \sqrt{\tau}) \equiv \ln(\frac{K}{S}) + \frac{\sigma^2}{2} \tau \).

• At any time \( t \in [0, T] \), we define the at-the-money (ATM) strike rate as the unique strike rate \( K^a_t > 0 \) that zeroes out log-moneyness:

\[
\ell_-(\frac{K^a_t}{S_t}, l_{at} \sqrt{T - t}) = 0,
\]

where \( l_{at} \equiv l_t(K^a_t) \) denotes the ATM IV quoted at the varying time \( t \geq 0 \) for the fixed maturity date \( T \geq t \).

• A straddle is a long position in one call and one put with the same underlying FX rate, strike rate, and maturity date.

• A straddle maturing at \( T > 0 \) is said to be ATM at the fixed time \( t \in [0, T] \) and fixed spot level \( S_t > 0 \) if the common strike rate of the put and call is \( K^a_t \).

• We allow a trader to continuously trade (always) ATM straddles. We also allow continuous trading in two (always) out-of-the-money (OTM) options.
Out-of-the-Money Put and Call

- For some time-varying strike rate $K_t^p < K_t^a$, let $I_{pt} \equiv I_t(K_t^p)$ be an (always) OTM put IV at time $t \in [0, T]$, defined so that:

$$P_t(K_t^p) = P_b(S_t, I_{pt}, T - t; K_t^p), \quad t \in [0, T].$$

- The BMS call value with fixed strike rate $K > 0$ and fixed maturity date $T > t \geq 0$ is given by:

$$C_b(S, \sigma, \tau; K) \equiv SN(-z_-(K/S, \sigma\sqrt{\tau})) - KN(-z_+(K/S, \sigma\sqrt{\tau})), \tau \equiv T - t.$$

- For some time-varying strike rate $K_t^c > K_t^a$, let $I_{ct} \equiv I_t(K_t^c)$ denote the (always) OTM call IV at time $t \in [0, T]$, defined so that:

$$C_t(K_t^c) = C_b(S_t, I_{ct}, T - t; K_t^c), \quad t \in [0, T].$$
Instantaneous Gains

- We define the instantaneous gain $g_{P_t}(K^p_t)$ on the OTM put by:
  \[
  g_{P_t}(K^p_t) \equiv \left[ dP^b(S_t, I_t(K), T - t; K) \right]_{K=K^p_t}, \quad t \in [0, T].
  \]

- Let $A^b(S, \sigma, \tau; K) = P^b(S, \sigma, \tau; K) + C^b(S, \sigma, \tau; K)$ be the BMS model value of a straddle and let $A_t(K^a_t) = A^b(S_t, I_t(K^a_t), T - t; K^a_t) = P_t(K^a_t) + C(K^a_t)$ be the ATM straddle value at time $t \in [0, T]$. We define the instantaneous gain $g_{A_t}(K^a_t)$ on the ATM straddle by:
  \[
  g_{A_t}(K^a_t) \equiv \left[ dA^b(S_t, I_t(K), T - t; K) \right]_{K=K^a_t}, \quad t \in [0, T].
  \]

- We also define the instantaneous gain $g_{C_t}(K^c_t)$ on the OTM call by:
  \[
  g_{C_t}(K^c_t) \equiv \left[ dC^b(S_t, I_t(K), T - t; K) \right]_{K=K^c_t}, \quad t \in [0, T].
  \]
Instantaneous Gain of a 3 Strike Option Portfolio

- Let $\eta^p_t$ be the number of OTM puts held, let $\eta^a_t$ be the number of ATM straddles held, and let $\eta^c_t$ be the number of OTM calls held at time $t \in [0, T]$.
- Let $V_t$ be the value of the following three strike rate option portfolio at time $t \in [0, T]$:
  \[ V_t \equiv \eta^p_t P_t(K^p_t) + \eta^a_t A_t(K^a_t) + \eta^c_t C_t(K^c_t), \quad t \in [0, T]. \]
- We define the instantaneous gain on this portfolio at $t \in [0, T]$ as:
  \[ gV_t \equiv \eta^p_t gP_t(K^p_t) + \eta^a_t gA_t(K^a_t) + \eta^c_t gC_t(K^c_t), \quad t \in [0, T]. \]
- Thus, the inst. gain $gV_t$ differs from the total derivative $dV_t$ in that the former suppresses the time variation of the option holdings $\eta$.
- As a result, the instantaneous gain in value of the option portfolio is just a linear combination of the previously defined instantaneous gains at each of the three traded strike rates.
Decomposing Inst. Gain of the OTM Put

• Recall that the instantaneous gain $g_{P_t}(K_t^p)$ on the OTM put is

$$g_{P_t}(K_t^p) \equiv \left[ dP^b(S_t, I_t(K), T - t; K) \right]_{K=K_t^p}, \quad t \in [0, T],$$

where under $Q_-$, the spot FX rate $S$ and the IV curve $I(K), K > 0$ solve the SDE’s:

$$dS_t = \sigma_t S_t dW_t, \quad dl_t(K) = \omega_t I_t(K) dZ_t, \quad d\langle W, Z \rangle_t = \rho_t dt.$$

• For $t \in [0, T]$, Itô’s formula implies that the instantaneous gain $g_{P_t}(K_t^p)$ on the OTM put decomposes as:

$$g_{P_t}(K_t^p) = P_1^b(S_t, I_{pt}, T - t; K_t^p) dS_t + P_2^b(S_t, I_{pt}, T - t; K_t^p) dl_{pt} + G_t^p dt,$$

where since $S$ and $I_p$ are $Q_-$ local martingales, $G_t^p$ is the mean gain rate under $Q_-$ on the OTM put at time $t \in [0, T]$, given on the next slide.
Mean Gain Rate of OTM Put

• Under $\mathbb{Q}_-$, the mean gain rate on the OTM put is:

$$
\mathcal{G}_t^p = P_{11}^b(S_t, l_{pt}, T - t; K^p_t) \frac{d\langle S \rangle_t}{2 dt} + P_{12}^b(S_t, l_{pt}, T - t; K^p_t) \frac{d\langle S, l_p \rangle_t}{dt} \\
+ P_{22}^b(S_t, l_{pt}, T - t; K^p_t) \frac{d\langle l_p \rangle_t}{2 dt} - P_3^b(S_t, l_{pt}, T - t; K^p_t).
$$

• Since $dS_t = \sigma_t S_t dW_t$, $dl_t(K) = \omega_t l_t(K) dZ_t$, and $d\langle W, Z \rangle_t = \rho_t dt$, the second order variations are:

$$
\frac{d\langle S \rangle_t}{dt} = \sigma^2_t S^2_t, \quad \frac{d\langle S, l_p \rangle_t}{dt} = S_t \sigma_t \rho_t \omega_t l_{pt} \equiv S_t \gamma_t l_{pt}, \quad \frac{d\langle l_p \rangle_t}{dt} = \omega_t l_{pt}^2.
$$

• Substituting implies that the $\mathbb{Q}_-$ mean gain rate on the OTM put is:

$$
\mathcal{G}_t^p = S_t^2 P_{11}^b \frac{\sigma^2_t}{2} + S_t l_{pt} P_{12}^b \gamma_t + l_{pt}^2 P_{22}^b \frac{\omega^2_t}{2} - P_3^b,
$$

with all greeks evaluated at $(S_t, l_{pt}, T - t; K^p_t)$. 
Mean Gain Rate of OTM Put (Con’d)

- Recall that the $Q_-$ mean gain rate on the OTM put is:

$$G_t^P = S_t^2 P_{11}^b \frac{\sigma_t^2}{2} + S_t I_{pt} P_{12}^b \gamma_t + I_{pt}^2 P_{22}^b \frac{\omega_t^2}{2} - P_3^b,$$

at $(S_t, I_{pt}, T - t; K_t^P)$.

- Hence, the $Q_-$ mean gain on the OTM put at time $t \in [0, T]$ is a linear combination of the put’s cash gamma: $\Gamma_t^P \equiv S_t^2 P_{11}^b$, its cash vanna: $S_t I_{pt} P_{12}^b = \Gamma_t^P \ell_{-t}^P, \ell_{-t}^P \equiv \ell_-(K_t^P / S_t, I_{pt} \sqrt{T - t})$, its cash volga: $I_{pt}^2 P_{22}^b = \Gamma_t^P \ell_{-t}^P \ell_{+t}^P, \ell_{+t}^P \equiv \ell_+(K_t^P / S_t, I_{pt} \sqrt{T - t})$, and its theta: $-P_3^b = -\Gamma_t^P I_{pt}^2 / 2$.

- Thus, the $Q_-$ mean gain rate on the OTM put simplifies to:

$$G_t^P = \Gamma_t^P \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t}^P + \frac{\omega_t^2}{2} \ell_{-t}^P \ell_{+t}^P - \frac{I_{pt}^2}{2} \right].$$
Inst. Gains of OTM Options & ATM Straddle

• The inst. gain on the OTM put at time $t \in [0, T]$ is given by:

$$g_{P_t}(K^p_t) = P^b_1(S_t, I_{pt}, T - t)\sigma_t S_t dW_t + P^b_2(S_t, I_{pt}, T - t)\omega_t I_{pt} dZ_t + \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t I^p_{-t} + \frac{\omega^2_t}{2} I^p_{-t} I^p_{+t} - \frac{I^2_{pt}}{2} \right] dt.$$ 

• The inst. gain on the OTM call is analogously given by:

$$g_{C_t}(K^c_t) = C^b_1(S_t, I_{ct}, T - t)\sigma_t S_t dW_t + C^b_2(S_t, I_{ct}, T - t)\omega_t I_{ct} dZ_t + \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t I^c_{-t} + \frac{\omega^2_t}{2} I^c_{-t} I^c_{+t} - \frac{I^2_{ct}}{2} \right] dt.$$ 

• The inst. gain on the ATM straddle at time $t \in [0, T]$ is simpler:

$$g_{A_t}(K^a_t) = A^b_1(S_t, I_{at}, T - t)\sigma_t S_t dW_t + A^b_2(S_t, I_{at}, T - t)\omega_t I_{at} dZ_t + \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] dt,$$

since vanna and volga vanish.
Inst. Gains of 3 Strike Option Portfolio

- The inst. gain on the 3 strike rate option portfolio is:

\[ gV_t \equiv \eta_t^p gP_t(K_t^p) + \eta_t^a gA_t(K_t^a) + \eta_t^c gC_t(K_t^c) \]

\[ = G_t^\nu dt + \Delta_t^\nu \sigma_t S_t dW_t + \$v_t^\nu \omega_t dZ_t, \]

where \( G_t^\nu \) is the \( \mathbb{Q} \)-mean gain rate on the option portfolio.

\[ G_t^\nu = \eta_t^p \Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t} + \frac{\omega_t^2}{2} \ell_{-t} \ell_{+t} - \frac{l_{pt}^2}{2} \right] + \eta_t^a \Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{l_{at}^2}{2} \right] \]

\[ + \eta_t^c \Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_{-t} + \frac{\omega_t^2}{2} \ell_{-t} \ell_{+t} - \frac{l_{ct}^2}{2} \right], \]

\( \Delta_t^\nu \) is the foreign currency delta of the option portfolio:

\[ \Delta_t^\nu = \eta_t^p P_t^b(S_t, l_{pt}, \tau) + \eta_t^a A_t^b(S_t, l_{at}, \tau) + \eta_t^c C_t^b(S_t, l_{ct}, \tau), \]

\( \tau \equiv T - t \), while \( \$v_t^\nu \) is the cash vega of the option portfolio:

\[ \$v_t^\nu \equiv \eta_t^p l_{pt} P_t^b(S_t, l_{pt}, \tau) + \eta_t^a l_{at} A_t^b(S_t, l_{at}, \tau) + \eta_t^c l_{ct} C_t^b(S_t, l_{ct}, \tau) \]

\[ = (T - t) \left[ \eta_t^p \Gamma_t^p l_{pt}^2 + \eta_t^a \Gamma_t^a l_{at}^2 + \eta_t^c \Gamma_t^c l_{ct}^2 \right]. \]
Focussing the Signal

• Recall that the inst. gain on the 3 strike rate option portfolio is:
  \[ gV_t = G^v_t dt + \Delta^v_t \sigma_t S_t dW_t + \$v^v_t \omega_t dZ_t, \quad t \in [0, T], \]
  where:
  \[
  G^v_t = \eta^p_t \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p_{-t} + \frac{\omega^2_t}{2} \ell^p_{-t} \ell^p_t - \frac{l^2_{pt}}{2} \right] + \eta^a_t \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{l^2_{at}}{2} \right]
  \]
  \[
  + \eta^c_t \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c_{-t} + \frac{\omega^2_t}{2} \ell^c_{-t} \ell^c_t - \frac{l^2_{ct}}{2} \right].
  \]

• Thus, the inst. gain is the sum of the signal \( G^v_t dt \) plus noise.

• In general, the signal depends on 3 stochastic processes, namely the inst. variance rate \( \sigma^2_t \) of \( \ln S \), the inst. covariation rate \( \gamma_t \equiv \sigma_t \rho_t \omega_t \) between \( \ln S \) and \( \ln I(K) \), and the inst. variance rate \( \omega^2_t \) of \( \ln I(K) \).

• When the signal is positive, we call the resulting option portfolio a **statistical arbitrage**.

• Suppose that a trader can only forecast 1 of the 3 stochastic processes. Can the signal just depend on the forecastable process w/o knowing the other 2 stochastic processes?
Vol Trade

- Recall that the $\mathbb{Q}_-$ mean gain rate is:

$$
G^\mathbb{Q}_t = \eta^p_t \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p_{-t} + \frac{\omega_t}{2} \ell^p_{-t} \ell^p_{+t} - \frac{I^2_{pt}}{2} \right] + \eta^a_t \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] + \eta^c_t \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c_{-t} + \frac{\omega_t}{2} \ell^c_{-t} \ell^c_{+t} - \frac{I^2_{ct}}{2} \right].
$$

- To avoid exposure to $\gamma_t$ and $\omega_t$, consider these option holdings:

$$
\eta^p_t = 0 \quad \eta^a_t = \frac{2}{\Gamma^a_t} \quad \eta^c_t = 0.
$$

- The position is always long $\frac{2}{\Gamma^a_t}$ units of an ATM straddle.

- As spot moves, the no longer ATM straddle must be sold and the freshly minted ATM straddle must be purchased.

- What is the $\mathbb{Q}_-$ mean gain rate on this trading strategy in ATM straddles?
Vol Trade (Con’d)

• Recall that the $\mathbb{Q}_-$ mean gain rate is:

$$G^\nu_v \equiv \eta^p_t \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p_t \ell^p_{-t} \ell^p_{+t} - \frac{I^2_{pt}}{2} \right] + \eta^a_t \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] + \eta^c_t \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c_t \ell^c_{-t} \ell^c_{+t} - \frac{I^2_{ct}}{2} \right].$$

• Also recall the so-called vol trade:

$$\eta^p_t = 0 \quad \eta^a_t = \frac{2}{\Gamma^a_t} \quad \eta^c_t = 0.$$

• Subbing into the top equation implies the following $\mathbb{Q}_-$ mean gain rate:

$$G^\nu_v \equiv \sigma^2_t - I^2_{at}.$$

• If a trader knows that all of the possible realizations of $\sigma_t$ lie above the ATM IV, then the vol trade is profitable on average.
Skew Trade

• We now suppose that $\ell^c_{\pm t} > 0$, $\ell^p_{\pm t} < 0$, and that the put and the call are equally OTM using geometric mean log moneyness:

$$\sqrt{\ell^p_{-t} \ell^p_{+t}} = \sqrt{\ell^c_{-t} \ell^c_{+t}} \equiv \bar{\ell}_{gt}, \quad t \in [0, T].$$

• In this case, the $\mathbb{Q}$ mean gain rate is:

$$G^\gamma_t = \eta^p_t \Gamma^p_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^p - \frac{I^2_{pt}}{2} \right] + \eta^a_t \Gamma^a_t \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right]$$

$$+ \eta^c_t \Gamma^c_t \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell^c - \frac{I^2_{ct}}{2} \right].$$

• To avoid exposure to $\sigma_t$ and $\omega_t$, consider these option holdings:

$$\eta^P_t = -\frac{1}{(\ell^c_{-t} - \ell^p_{-t})\Gamma^p_t}, \quad \eta^a_t = 0, \quad \eta^c_t = \frac{1}{(\ell^c_{-t} - \ell^p_{-t})\Gamma^c_t},$$

viz $\frac{1}{\ell^c_{-t} - \ell^p_{-t}}$ normalized risk-reversals, each valued at $\frac{C_t(K^c_t)}{\Gamma^c_t} - \frac{P_t(K^p_t)}{\Gamma^p_t}$.

• As $S$ moves, both OTM options must be traded.
Skew Trade (Con’d)

• Recall the $\mathbb{Q}_-$ mean gain rate when the put and the call are equally OTM using geometric mean log moneyness:

$$G_t^\gamma = \eta_t^p \Gamma_t^p \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_t^p + \frac{\omega_t^2}{2} \ell_{gt}^2 - \frac{I_{pt}^2}{2} \right] + \eta_t^a \Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{I_{at}^2}{2} \right] + \eta_t^c \Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_t^c + \frac{\omega_t^2}{2} \ell_{gt}^2 - \frac{I_{ct}^2}{2} \right],$$

and the skew trade: $\eta_t^p = -\frac{1}{(\ell^c_t - \ell^p_t)\Gamma_t^p}, \eta_t^a = 0, \eta_t^c = \frac{1}{(\ell^c_t - \ell^p_t)\Gamma_t^c}$.

• Subbing into the top eq’n implies that this skew trade has the following $\mathbb{Q}_-$ mean gain rate in the value of the position:

$$G_t^\gamma \equiv \gamma_t - \frac{I_{ct}^2}{2} - \frac{I_{pt}^2}{2}. $$

• When all of the possible realizations of $\gamma_t$ lie above the halved implied variance slope, the skew trade is profitable on average.
Smile Trade

• Suppose that the call and put are equally OTM using the \( \ell_- \) measure of log moneyness: \( \ell_{-t}^c = -\ell_{-t}^p \equiv \ell_- > 0, \ t \in [0, T] \).

• The \( \mathbb{Q}_- \) mean gain rate of the option portfolio simplifies slightly to:

\[
G_t^\nu = \eta_t^p \Gamma_t^p \left[ \frac{\sigma_t^2}{2} - \gamma_t \ell_- - \frac{\omega_t^2}{2} \ell_- \ell_+ - \frac{I_{pt}^2}{2} \right] + \eta_t^a \Gamma_t^a \left[ \frac{\sigma_t^2}{2} - \frac{I_{at}^2}{2} \right] + \eta_t^c \Gamma_t^c \left[ \frac{\sigma_t^2}{2} + \gamma_t \ell_- + \frac{\omega_t^2}{2} \ell_- \ell_+ - \frac{I_{ct}^2}{2} \right].
\]

• To avoid exposure to \( \sigma \) and \( \gamma \), consider these option holdings:

\[
\eta_t^p = \frac{1}{\ell_{agt}^2 \Gamma_t^p} \quad \eta_t^a = -\frac{2}{\ell_{agt}^2 \Gamma_t^a} \quad \eta_t^c = \frac{1}{\ell_{agt}^2 \Gamma_t^c} \quad t \in [0, T],
\]

where \( \ell_{agt} \equiv \sqrt{\frac{\ell_{-t}^c + |\ell_{-t}^p|}{2} \frac{\ell_+^c + |\ell_+^p|}{2}} \).

• This is a long position of \( \frac{1}{\ell_{agt}^2} \) normalized butterflies, each with value

\[
\frac{C_t(K_t^c)}{\Gamma_t^c} - 2 \frac{A_t(K_t^a)}{\Gamma_t^a} + \frac{P_t(K_t^p)}{\Gamma_t^p}.
\]
Smile Trade (Con’d)

- Recall the $\mathbb{Q}_-$ mean gain rate of the option portfolio when the call and put are equally OTM using the $\ell_-$ measure of log moneyness:

  \[
  G_v^t = \eta_t^p \Gamma_t^p \left[ \frac{\sigma^2_t}{2} - \gamma_t \ell_{-t} - \frac{\omega^2_t}{2} \ell_{-t} \ell_{+t} - \frac{I^2_{pt}}{2} \right] + \eta_t^a \Gamma_t^a \left[ \frac{\sigma^2_t}{2} - \frac{I^2_{at}}{2} \right] + \eta_t^c \Gamma_t^c \left[ \frac{\sigma^2_t}{2} + \gamma_t \ell_{-t} + \frac{\omega^2_t}{2} \ell_{-t} \ell_{+t} - \frac{I^2_{ct}}{2} \right].
  \]

- Also recall the smile trade:

  \[
  \eta_t^p = \frac{1}{\ell_{agt}^2 \Gamma_t^p}, \quad \eta_t^a = -\frac{2}{\ell_{agt}^2 \Gamma_t^a}, \quad \eta_t^c = \frac{1}{\ell_{agt}^2 \Gamma_t^c}, \quad t \in [0, T].
  \]

- Substituting into the top equation implies that this smile trade has:

  \[
  G_v^t = \omega^2_t - \frac{l^2_{ct} + l^2_{pt}}{\ell_{agt}^2} - \frac{l^2_{at}}{2}.
  \]

- When all possible realizations of $\omega^2_t$ lie above the convexity measure of halved implied variance, the smile trade is profitable on average.
Summary

- We assumed $dS_t = \sigma_t S_t dW_t$, $dl_t(K) = \omega_t l_t(K) dZ_t$, for $K > 0$, where $W, Z$ are $\mathbb{Q}$-SBM's with $\rho_t \in [-1, 1]$, $t \in [0, T]$.

- The inst. vol $\sigma_t > 0$, the vol-vol $\omega_t > 0$, and the correlation $\rho_t \in [-1, 1]$ are all unspecified stochastic processes.

- By dynamically trading ATM straddles, normalized risk reversals, or normalized butterfly spreads, a trader was able to synthesize a short-term forward contract on $\sigma_t^2$, on $\gamma_t \equiv \sigma_t \rho_t \omega_t$, or on $\omega_t^2$.

- The forward price paid for each of these three bets was either $l_{at}^2$ (vol trade), $\frac{l_{ct}^2 - l_{pt}^2}{\ell_c^2 - \ell_p^2}$ (skew trade), or $\frac{l_{ct}^2 + l_{pt}^2}{\ell_{agt}^2} - l_{at}^2$ (smile trade).

- The three forward prices measure level, slope, and convexity of a (halved) implied variance curve.
Summary (Con’d)

- Our paper gives financial meaning to measures of level, slope and curvature of halved implied variances. However, the standard FX option quotes are in terms of implied vol’s, not halved implied variances.

- The paper shows that the three standard FX option quotes are positive multiples of the vega of normalized ATM straddles, normalized risk-reversals, and normalized butterfly-spreads.

- This presentation gave sufficient conditions under which the three IV quotes produce statistical arbitrage. However, the paper also gives sufficient conditions under which the three IV quotes produce riskless arbitrage. The exact positions needed to realize this arbitrage are also given.

- Thanks for listening!