52nd IEEE Conference on Decision and Control
December 10-13, 2013. Florence, Italy

Price and Variance of Anarchy in Mean-Variance Cost Density-Shaping Stochastic Differential Games

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***Abstract*— This paper introduces the Variance of Anarchy (VoA) metric to compliment existing measures of efficiency loss in dynamic games due to decentralized mechanisms. The VoA is inspired by Price of Anarchy (PoA) and Price of Information (PoI) measures that have been used previously in the literature. We propose a new design procedure for decentralized control algorithms using PoA and VoA that identifies the optimal control solution for competing agents among a family of decentralized controllers by solving an optimization of a summed PoA and VoA objective function over a parameter space. The design method is illustrated with a stochastic model for queue server dynamics and two separate optimal control problems - the first involving noncooperative agents, and the second a team. For each problem, a family of density-shaping cumulant controls is computed corresponding to a parametric target cumulant set, and the optimal chosen via this new procedure. Simulation results are provided to compare the controller to a baseline $2CC$ control.**

***Index Terms*— Cost Density-Shaping Games, Cost Cumulant Control, Price of Anarchy, Variance of Anarchy, Telecommunications, Stochastic Differential Games, Team Optimization.**

I. INTRODUCTION

In the past decade, statistical games involving cost cumulant control have made their debut in the research community for controls and decision theory. Unlike classical stochastic control theory, cost cumulant control seeks to optimize high-order statistics of the random cost functional beyond its mean, such as the cost variance. This feature of cost cumulant control has led to performance gains in statistical games because of variance optimization of random payoffs belonging to competing agents. The types of statistical games that have been considered in the literature range from purely noncooperative games with the Nash equilibrium solution concept, to nonzero-sum or min-max games, and games involving cooperative teams of agents where competition exists amongst teams [4].

Cost-density shaping theory has been added to the existing control paradigms which collectively embody statistical control [4], [5], [6]. The aim of cost density-shaping is to find a linear control that achieves a target shape for the probability density of the random cost functional. This target shape is captured by finite set of target cost cumulants that tightly constrain the cost density, and cost density-shaping controllers basically track these target statistics on the time horizon to achieve density-shaping objectives. Recently, the cost density-shaping theory has been extended to N-player Nash games [5] with promising results in structural control that motivate continued development of the theory.

In this paper, we introduce metrics to compare the Nash equilibrium solution of an N-person stochastic differential game under cost-density shaping with the centralized optimal control solution for the social welfare problem associated with the game. One of such metrics is Price of Anarchy (PoA), which has been used to measure the efficiency loss of Nash equilibrium with respect to the optimal solution of its associated team problem. In the context of cost density-shaping methods, we define PoA as ratio between the mean of optimal team cost and the sum of the means of player costs under Nash equilibrium. Inspired by PoA measure of loss of efficiency, we also propose the metric of Variance of Anarchy (VoA) to measure the variance on efficiency loss in a control problem from noncooperation among N agents. Its definition involves the variance of the random cost instead of the mean.

The concept of PoA has gained a significant amount of interest in the past few years [10], [11], [12], [7], [8], [9]. It serves as a metric to quantify the efficiency loss of decentralized mechanisms, and also provides a criterion for designing efficient mechanisms. Most applications of PoA so far are limited to static games, such as auction games [13], continuous-kernel static games [10], etc. The applications have been in e-commerce [12], market design [14], internet congestion control [15], etc. However, there has been very few work that extends this notion to dynamic games or dynamic mechanism design problems.

Our work is closely related to [7], [8]. In [7], [8], the authors have made extensions of the concept of PoA to a general differential games framework. In addition, price of information (PoI) is introduced to compare game performances under different information structures. Bounds and large-population approximations on the PoI and the PoA are obtained for feedback differential games. To the best knowledge of the authors, this is the first work to apply PoA and VoA measures to assess the efficacy of noncooperation in statistical games, and use them for design purposes.

Six major sections comprise this paper. In the first, the model inspiring the two separate problems posed in this work is given. The second section provides the problem

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formulation for the noncooperative problem and the control solution. The third section is analogous to the second section, but focuses on the team problem. The fourth section contains the definitions for PoA and VoA that give the price of noncooperation in cost density-shaping games. Following this, the fifth section poses a PoA-VoA optimization for control design.The sixth section contains an telecommunications application where efficacy of noncooperation is assessed on a network with a queuing server and two network users, and the new optimization method is used. A summary and concluding remarks follow.

II. System Model

We adopt here the communication systems model described in [1], [2], where the players are the *users* or *sources*, and the action (control) variables are the *flows* into the network. If a link receives more total flow than what it can accommodate (measured by its capacity), then packets queue up. Having long queues is not desirable, because it leads to delays in transmission. We call such links which are congested *bottleneck links*, and formulate the game around one such link. Let $q_l(t)$ denote the queue length at such a bottleneck link and let $s_r(t)$ denote the total effective service rate available at that link.

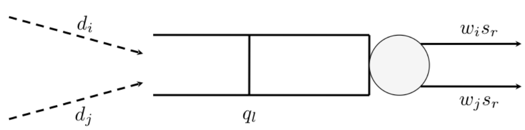

Fig. 1: Illustration of the queueing model: players i and j send data packets to the queue at rate d_i and d_j, respectively, and they are serviced by the queue at the rate $w_i s_r$ and $w_j s_r$, respectively, in order to regulate the queue length q_l at a certain level.

Assume that each user is assigned a fixed proportion of the available bandwidth; more specifically, the traffic of source $i, i = 1,2,\ldots,N$, has an allotted bandwidth of $w_i s(t)$, where w_i's are positive parameters which add up to 1. We assume that the users have perfect measurement of $s(t)$, but occasionally exceed or fall short of the bandwidth allotted to them due to fluctuations. Hence, if $d_i(t)$ denotes the rate of source i at time t, we can introduce $u_i(t) := d_i(t) - w_i s_r(t)$ as the control (action) variable of the source. Then, queue build-up is governed by the differential equation

$$\dot{q}_l(t) = \sum_{i=1}^{N} u_i(t)\,, \tag{1}$$

where we assume that queue is relatively tightly controlled so that end effect constraints (starvation and exceeding an upper limit) do not become active. The goal is to ensure that the bottleneck queue size stays around some desired level $\bar{q}_l$, and good tracking between input and output rates is achieved. Toward that end, we consider the shifted variable $x(t) := q_l(t) - \bar{q}_l$, which satisfies the following differential equation which is the shifted version of (1):

$$\dot{x}(t) = \sum_{i=1}^{N} u_i(t) \quad x(0) = x_0\,. \tag{2}$$

The model (2) can be extended to include additional random network phenomena, such as dropped packets and demand surges. We consider a stochastic version of the original model with additive white noise

$$dx = \sum_{i=1}^{N} u_i(t)dt + \mathbf{1}dw(t),\ x_0 = \mathbb{E}_{\mathscr{P}_0}\{x(t_0)\} \tag{3}$$

Above $x \in \mathbb{R}^n$, $u_i \in \mathbb{R}^n$, $\mathbf{1}^T = [1\ldots1] \in \mathbb{R}^n$ and $w(t)$ is a one-dimensional Wiener process with correlation of increments

$$E[(w(\tau_1) - w(\tau_2))(w(\tau_1) - w(\tau_2))^T] = W|\tau_1 - \tau_2|.$$

We use (3) to motivate two particular problems: the noncooperative problem and the team (or social welfare) problem. We pose each problem separately, and setup a framework for cost density-shaping.

III. Noncooperative Problem

This section is intended to establish the noncooperative stochastic optimal control problem and pose a mean-variance cost density-shaping optimization for it. We present the process model, the cost function, its cumulant dynamics equations, the target cumulants characterizing a family of target cost densities for this cost, and finally a cost density-shaping problem and its control solution.

***Definition** 3.1 (Stochastic Game Process):* Consider the scalar stochastic differential equations given by

$$dx = \frac{1}{N}\sum_{i=1}^{N} u_i(t)dt + \mathbf{1}dw(t),\ x_0 = \mathbb{E}_{\mathscr{P}_0}\{x(t_0)\}, x \in \mathbb{R}. \tag{4}$$

Suppose that in the *noncooperative case*, each network user chooses his or her demand to optimize the statistical characterization of a random cost below.

***Definition** 3.2 (Cost Functionals):* Consider the integral-quadratic form given below as the cost for Player i

$$J_i(u_i) = \int_{t_0}^{t_f} \left(\frac{1}{N}|x(t)|^2 + \frac{1}{c_i}|u_i(t)|^2\right)dt + s_i(x(t_f)) \tag{5}$$

Suppose that control inputs are restricted to linear, state-feedback, memoryless control laws, i.e.,

$$u_i(t) = k_i(t)x(t),\ k_i(t) \in \mathbb{R},\ t \in [t_0, t_f],\ i = 1,2. \tag{6}$$

In the proceeding, we further assume that $n = 1$, $N = 2$, $s_1(x) = s_2(x) = x^2$, $W = 1$, and $c_1 = c_2 = 2$ to write the equations for the cost cumulants of (5) using the theoretical framework presented in [5].

***Definition** 3.3 (Cost Cumulants):* Given the process (4), the cost (5), and the control (6), a stochastic optimal control problem for the noncooperative case fits the LQG framework. When $\alpha = t_f$, the initial mean and variance of $J_i(u_i)$ are

$$\begin{aligned}\mathbb{E}\{J_j|x(t_0) = x_0\} = \kappa_1^j(\alpha) = h_1^j(\alpha)x_0^2 + d_1^j(\alpha),\\ \mathrm{Var}\{J_j|x(t_0) = x_0\} = \kappa_2^j(\alpha) = h_2^j(\alpha)x_0^2 + d_2^j(\alpha),\end{aligned} \tag{7}$$

where the variables $h_i^j(\alpha), d_i^j(\alpha)$ for $j=1,2$ and $i=1,2$ satisfy the backwards differential equations

$$\begin{aligned}
&\frac{dh_1^j(\alpha)}{d\alpha} = -(k_1(\alpha)+k_2(\alpha))h_1^j(\alpha) - \frac{1}{2}(k_j(\alpha))^2 - \frac{1}{2},\\
&\frac{dh_2^j(\alpha)}{d\alpha} = -(k_1(\alpha)+k_2(\alpha))h_2^j(\alpha) - 4(h_1^j(\alpha))^2,\\
&\frac{d}{d\alpha}d_1^j(\alpha) = h_1^j(\alpha), \frac{d}{d\alpha}d_2^j(\alpha) = h_2^j(\alpha), \alpha \in [t_0,t_f],\\
&h_1^j(t_f)=1, h_2^j(t_f)=0, d_1^j(t_f)=0, d_2^j(t_f)=1.
\end{aligned} \tag{8}$$

It will be convenient to write the equations of motion for the $h_i^j(\alpha)$ variables above in standard form $\dot{x} = f(x,u)$,

$$\begin{aligned}
&\dot{h}_1^j(\alpha) = f_1(h_1^j(\alpha), k_1(\alpha), k_2(\alpha)), h_1^j(t_f)=1,\\
&\dot{h}_2^j(\alpha) = f_2(h_1^j(\alpha), h_2^j(\alpha), k_1(\alpha), k_2(\alpha)), h_2^j(t_f)=0,
\end{aligned}$$

where $f_1(\cdot), f_2(\cdot)$ have obvious definitions from (8). Using this notation, we now define the target cumulants for the noncooperative case.

***Definition** 3.4:* (Target Cumulants) Let the target cost cumulants for the noncooperative problem be

$$\tilde{\kappa}_i^j(\alpha,\theta) = (1-\theta)\tilde{\kappa}_{i,LQG}^j(\alpha) + \theta\tilde{\kappa}_{i,2CC}^j(\alpha), \tag{9}$$

for players $j=1,2$ with $0 \leq \theta \leq 1$. The quantities $\tilde{\kappa}_{i,LQG}^j(\alpha)$ and $\tilde{\kappa}_{i,2CC}^j(\alpha)$ are given by

$$\begin{aligned}
&\tilde{\kappa}_{i,LQG}^j(\alpha) = \tilde{h}_{i,LQG}^j(\alpha)x_0^2 + \tilde{d}_{i,LQG}^j(\alpha),\\
&\tilde{\kappa}_{i,2CC}^j(\alpha) = \tilde{h}_{i,2CC}^j(\alpha)x_0^2 + \tilde{d}_{i,2CC}^j(\alpha), i=1,2,
\end{aligned}$$

with the variables $\tilde{h}_{i,LQG}^j(\alpha)$ and $\tilde{d}_{i,LQG}^j(\alpha)$ determined by

$$\begin{aligned}
&\dot{\tilde{h}}_{1,LQG}^j(\alpha) = f_1(\tilde{h}_{1,LQG}^j(\alpha), \tilde{k}_{1,LQG}(\alpha), \tilde{k}_{2,LQG}(\alpha)),\\
&\dot{\tilde{h}}_{2,LQG}^j(\alpha) =\\
&\qquad f_2(\tilde{h}_{1,LQG}^j(\alpha), \tilde{h}_{2,LQG}^j(\alpha), \tilde{k}_{1,LQG}(\alpha), \tilde{k}_{2,LQG}(\alpha)),\\
&\dot{\tilde{d}}_{i,LQG}^j(\alpha) = \tilde{h}_{i,LQG}^j(\alpha),\ \tilde{h}_{1,LQG}^j(t_f)=1, \tilde{h}_{2,LQG}^j(t_f)=0,\\
&\tilde{d}_{1,LQG}^j(t_f) = 1.0\times 10^{-9}, \tilde{d}_{2,LQG}^j(t_f)=1,
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\tilde{\kappa}_{i,2CC}^j(\alpha) = \tilde{h}_{i,2CC}^j(\alpha)x_0^2 + \tilde{d}_{i,2CC}^j(\alpha), i=1,2,\\
&\dot{\tilde{h}}_{1,2CC}^j(\alpha) = f_1(\tilde{h}_{1,2CC}^j(\alpha), \tilde{k}_{1,2CC}(\alpha), \tilde{k}_{2,2CC}(\alpha)),\\
&\dot{\tilde{h}}_{2,2CC}^j(\alpha) =\\
&\qquad f_2(\tilde{h}_{1,2CC}^j(\alpha), \tilde{h}_{2,2CC}^j(\alpha), \tilde{k}_{1,2CC}(\alpha), \tilde{k}_{2,2CC}(\alpha)),\\
&\dot{\tilde{d}}_{i,2CC}^j(\alpha) = \tilde{h}_{i,2CC}^j(\alpha),\ \tilde{h}_{1,2CC}^j(t_f)=1, \tilde{h}_{2,2CC}^j(t_f)=0,\\
&\tilde{d}_{1,2CC}^j(t_f) = 1.0\times 10^{-9}, \tilde{d}_{2,2CC}^j(t_f)=1.
\end{aligned} \tag{11}$$

These differential equations are characterized by the target controllers

$$\begin{aligned}
&\tilde{k}_{j,LQG}(\alpha) = -\tilde{h}_{1,LQG}^j(\alpha),\\
&\tilde{k}_{j,2CC}(\alpha) = -\left(\tilde{h}_{1,2CC}^j(\alpha) + \mu\cdot\tilde{h}_{2,2CC}^j(\alpha)\right),\ \mu > 0.
\end{aligned}$$

Remark The 2CC optimal control problem identifies the optimal linear control gain to minimize the objective function $\mathbb{E}\{J\} + \mu\cdot Var\{J\}$. For well-posedness, $\mu > 0$ is required.

The cost density-shaping problem for the noncooperative case will involve normalized cost and target cost variates defined as below.

***Definition** 3.5 (Normalized Cost/Target Cost Variates):* Define normalized cost and target cost variates $Z_j, \tilde{Z}_j$ as

$$\begin{aligned}
&Z_j = \frac{J_j - \kappa_1^j(t_0)}{\kappa_2^j(t_0)^{1/2}},\ \tilde{Z}_j = \frac{J_j - \tilde{\kappa}_1^j(t_0,\theta)}{\tilde{\kappa}_2^j(t_0,\theta)^{1/2}} = a_j Z_j + b_j\\
&\tilde{Z}_j = \underbrace{\left(\frac{\kappa_2^j(t_0)}{\tilde{\kappa}_2^j(t_0,\theta)}\right)^{1/2}}_{a_j} \cdot \underbrace{\frac{J-\kappa_1^j(t_0)}{\kappa_2^j(t_0)^{1/2}}}_{Z_j} + \underbrace{\frac{\kappa_1^j(t_0) - \tilde{\kappa}_1^j(t_0,\theta)}{\tilde{\kappa}_2^j(t_0,\theta)^{1/2}}}_{b_j}
\end{aligned}$$

and let p_{Z_j} and $p_{\tilde{Z}_j}$ represent their best Gaussian density approximations,

$$p_{Z_j}(z) \approx \frac{1}{\sqrt{2\pi}}\cdot\exp\left(\frac{-z^2}{2}\right),\ p_{\tilde{Z}_j}(\tilde{z}) \approx a_j p_{Z_j}(a_j z + b_j).$$

Now we propose the Kullback-Leibler Divergence (*KLD*) for cost density-shaping, which has been used for density-shaping with good results. See [16] and references therein.

***Definition** 3.6 (Probability Distance Measure):* The Kullback-Leibler divergence between the best Gaussian density approximations to normalized cost and target cost variates Z_j and $\tilde{Z}_j$ respectively is

$$\begin{aligned}
KLD(p_{Z_j}(z), p_{\tilde{Z}_j}(z)) &= \int_{-\infty}^{\infty} p_{Z_j}(z)\log\left(\frac{p_{Z_j}(z)}{p_{\tilde{Z}_j}(z)}\right)dz\\
&= \frac{1}{2}\left(\frac{\kappa_2^j(t_0)}{\tilde{\kappa}_2^j(t_0,\theta)} - 1 - \log\left(\frac{\kappa_2^j(t_0)}{\tilde{\kappa}_2^j(t_0,\theta)}\right)\right)\\
&+ \frac{1}{2}\left(\frac{(\kappa_1^j(t_0) - \tilde{\kappa}_1^j(t_0,\theta))^2}{\tilde{\kappa}_2^j(t_0,\theta)}\right)\\
&= g\left(\begin{bmatrix}\kappa_1^j(t_0)\\ \kappa_2^j(t_0)\end{bmatrix}, \begin{bmatrix}\tilde{\kappa}_1^j(t_0,\theta)\\ \tilde{\kappa}_2^j(t_0,\theta)\end{bmatrix}\right).
\end{aligned}$$

We are now ready to pose the mean-variance cost density-shaping control problem involving $KLD(\cdot)$ for the noncooperative problem.

***Definition** 3.7:* (Noncooperative Cost-Density Shaping) Consider the following cost density-shaping optimization problem for Player $j=1,2$,

$$\begin{aligned}
&\min_{\kappa_1^j,\kappa_2^j}\{KLD(p_{Z_j}(z), p_{\tilde{Z}_j}(z))\}\\
&\text{subject to: eq. of motion (8), (10), (11).}
\end{aligned} \tag{12}$$

The control solution to the problem is given below.

***Theorem** 3.8 (Noncooperative Game Solution):* The optimal control solving (12) for Player j takes the form

$$k_j^*(\alpha,\theta) = -2\left(h_1^j(\alpha) + \left(\frac{\partial g(\cdot)}{\partial\kappa_2^j(\alpha)} \Big/ \frac{\partial g(\cdot)}{\partial\kappa_1^j(\alpha)}\right)h_2^j(\alpha)\right). \tag{13}$$

IV. Team Problem

Analogous to the section dedicated to the noncooperative problem, this section is intended to establish the team stochastic optimal control problem and pose a mean-variance cost density-shaping optimization for it.

***Definition** 4.1 (Stochastic Team Process):* Consider the stochastic differential equations given by

$$dx_T = u(t)dt + \mathbf{1}dw(t),\ x_0 = \mathbb{E}_{\mathscr{P}_0}\{x_T(t_0)\}. \tag{14}$$

Remark In the team formulation, we have assumed that all players have equal influence on the process evolution and that all players cooperate and maintain the same control input $u(t)$. Note that when $u_i(t) = u(t)$ in (4), then $x_T(t) = x(t)$.

Suppose that in the *team case*, one control input u is chosen for all network users to optimize the statistical characterization of the following team random cost.

***Definition** 4.2 (Team Cost Functionals):* Consider the integral-quadratic form given below as the

$$J_T(u) = \int_{t_0}^{t_f}\left(|x_T(t)|^2 + \frac{c_1+c_2}{c_1c_2}|u(t)|^2\right)dt + s(x_T(t_f)). \tag{15}$$

Suppose that control input is restricted to linear, state-feedback, memoryless control laws

$$u(t) = k(t)x_T(t),\ t\in[t_0,t_f]. \tag{16}$$

In the proceeding, we further assume that $n=1$, $N=2$, $s(x)=2x^2$, $c_1=c_2=2$, and $W=1$ to write the equations for the cost cumulants of (15).

***Definition** 4.3 (Cost Cumulants):* Given the process (14), the cost (15), and the control (16), a stochastic optimal control problem for the team case fits the LQG framework. When $\alpha = t_f$, the initial mean and variance of $J_i(u_i)$ are

$$\begin{aligned}\mathbb{E}\{J_T|x(t_0)=x_0\} &= \kappa_1^T(\alpha) = h_1^T(\alpha)x_0^2 + d_1^T(\alpha),\\ \mathrm{Var}\{J_T|x(t_0)=x_0\} &= \kappa_2^T(\alpha) = h_2^T(\alpha)x_0^2 + d_2^T(\alpha),\end{aligned} \tag{17}$$

where the variables $h_i^T(\alpha), d_i^T(\alpha)$ for $i=1,2$ satisfy the backwards differential equations

$$\begin{aligned}&\frac{dh_1^T(\alpha)}{d\alpha} = -2k(\alpha)h_1^T(\alpha) - (k(\alpha))^2 - 1,\\ &\frac{dh_2^T(\alpha)}{d\alpha} = -2k(\alpha)h_2^T(\alpha) - 4(h_1^T(\alpha))^2,\\ &\frac{d}{d\alpha}d_1^T(\alpha) = h_1^T(\alpha), \frac{d}{d\alpha}d_2^T(\alpha) = h_2^T(\alpha), \alpha\in[t_0,t_f],\\ &h_1^T(t_f)=1, h_2^T(t_f)=0, d_1^T(t_f)=0, d_2^T(t_f)=1.\end{aligned} \tag{18}$$

It will be convenient to write the equations of motion for the $h_i^T(\alpha)$ variables above in standard form $\dot{x} = f(x,u)$,

$$\begin{aligned}\dot{h_1^T}(\alpha) &= f_1^T(h_1^T(\alpha),k(\alpha)), h_1^T(t_f)=1,\\ \dot{h_2^T}(\alpha) &= f_2^T(h_1^T(\alpha),h_2^T(\alpha),k(\alpha)), h_2^T(t_f)=0.\end{aligned}$$

where $f_1^T(\cdot), f_2^T(\cdot)$ have obvious definitions from (18).

***Definition** 4.4 (Target Cumulants):* Let the target cost cumulants for the team problem be

$$\tilde{\kappa}_i^j(\alpha,\theta) = (1-\theta)\tilde{\kappa}_{i,LQG}^j(\alpha) + \theta\tilde{\kappa}_{i,2CC}^j(\alpha) \tag{19}$$

for players $j=1,2$ with $0\le\theta\le 1$. The quantities $\tilde{\kappa}_{i,LQG}^j(\alpha)$ and $\tilde{\kappa}_{i,2CC}^j(\alpha)$ are given by

$$\tilde{\kappa}_{i,LQG}^T(\alpha) = \tilde{h}_{i,LQG}^T(\alpha)x_0^2 + \tilde{d}_{i,LQG}^T(\alpha),$$

$$\tilde{\kappa}_{i,2CC}^T(\alpha) = \tilde{h}_{i,2CC}^T(\alpha)x_0^2 + \tilde{d}_{i,2CC}^T(\alpha), i=1,2.$$

with the variables $\tilde{h}_{i,LQG}^T(\alpha)$ and $\tilde{d}_{i,LQG}^T(\alpha)$ determined by

$$\begin{aligned}&\dot{\tilde{h}}_{1,LQG}^T(\alpha) = f_1(\tilde{h}_{1,LQG}^T(\alpha),\tilde{k}_{LQG}(\alpha)),\\ &\dot{\tilde{h}}_{2,LQG}^T(\alpha) = f_2(\tilde{h}_{1,LQG}^T(\alpha),\tilde{h}_{2,LQG}^T(\alpha),\tilde{k}_{LQG}(\alpha)),\\ &\dot{\tilde{d}}_{i,LQG}^T(\alpha) = \tilde{h}_{i,LQG}^T(\alpha),\ \tilde{h}_{1,LQG}^T(t_f)=1, \tilde{h}_{2,LQG}^T(t_f)=0,\\ &\tilde{d}_{1,LQG}^T(t_f) = 1.0\times10^{-9}, \tilde{d}_{2,LQG}^T(t_f)=1,\end{aligned} \tag{20}$$

and

$$\begin{aligned}&\dot{\tilde{h}}_{1,2CC}^T(\alpha) = f_1(\tilde{h}_{1,2CC}^T(\alpha),\tilde{k}_{2CC}(\alpha)),\\ &\dot{\tilde{h}}_{2,2CC}^T(\alpha) = f_2(\tilde{h}_{1,2CC}^T(\alpha),\tilde{h}_{2,2CC}^T(\alpha),\tilde{k}_{2CC}(\alpha)),\\ &\dot{\tilde{d}}_{i,2CC}^T(\alpha) = \tilde{h}_{i,2CC}^T(\alpha),\ \tilde{h}_{1,2CC}^T(t_f)=1, \tilde{h}_{2,2CC}^T(t_f)=0,\\ &\tilde{d}_{1,2CC}^T(t_f) = 1.0\times10^{-9}, \tilde{d}_{2,2CC}^T(t_f)=1.\end{aligned} \tag{21}$$

These differential equations are characterized by the target controllers

$$\begin{aligned}\tilde{k}_{LQG}(\alpha) &= -\tilde{h}_{1,LQG}^T(\alpha),\\ \tilde{k}_{2CC}(\alpha) &= -\left(\tilde{h}_{1,2CC}^T(\alpha) + \mu\cdot\tilde{h}_{2,2CC}^T(\alpha)\right),\ \mu>0.\end{aligned}$$

***Definition** 4.5 (Normalized Cost/Target Cost Variates):* Define normalized cost and target cost variates $Z_T, \tilde{Z}_T$ as

$$Z_T = \frac{J_T - \kappa_1^T(t_0)}{\kappa_2^T(t_0)^{1/2}},\ \tilde{Z}_T = a_TZ_T + b_T.$$

and let p_{Z_T} and $p_{\tilde{Z}_T}$ represent their best Gaussian density approximations,

$$p_{Z_T}(z) \approx \frac{1}{\sqrt{2\pi}}\cdot\exp\left(\frac{-z^2}{2}\right),\ p_{\tilde{Z}_T}(\tilde{z}) \approx a_Tp_{Z_T}(a_Tz+b_T).$$

Just as with the noncooperative problem, we will use the *KLD* for cost density-shaping. We can now pose the team cost density-shaping optimization.

***Definition** 4.6 (Team Cost-Density Shaping):* Consider the following cost density-shaping optimization problem for the team,

$$\begin{aligned}&\min_{\kappa_T}\{KLD(p_{Z_T}(z), p_{\tilde{Z}_T}(z))\}\\ &\text{subject to: eq. of motion (18), (20), (21).}\end{aligned} \tag{22}$$

The control solution to the problem is given in the below.

***Theorem** 4.7 (Team Control Solution):* The optimal control to the optimization (12) for the team takes the form

$$k^*(\alpha,\theta) = -\left(h_1^T(\alpha) + \left(\frac{\partial g(\cdot)}{\partial\kappa_2^T(\alpha)}/\frac{\partial g(\cdot)}{\partial\kappa_1^T(\alpha)}\right)h_2^T(\alpha)\right). \tag{23}$$

V. Measures of Noncooperation

The Price of Anarchy (PoA) is a measure of efficiency loss in a control problem due to noncooperation. The PoA can be defined as

$$\mathrm{PoA} \triangleq \frac{\mathbb{E}_{\mathscr{P}_{J_T}}\{J_T(u^*)\}}{\sum_{i=1}^N \mathbb{E}_{\mathscr{P}_{J_i}}\{J_i(u_i^*)\}}. \tag{24}$$

Essentially, the PoA can be viewed as a ratio of the team cost's mean to the sum of the means of player costs.

This measure should be bounded below and above, that is $0 < \text{PoA} < 1$. The PoA provides a good representation of how much performance is lost due to noncooperation only when J_T and $\sum_{i=1}^{N} J_i$ are directly comparable in terms of the weighting matrices and terminal penalty functions chosen. Furthermore, the designation of optimality for the control inputs u^*, u_i^*, and u_{N-i}^* above means that the controllers solve similar optimizations that separately involve the team cost and player costs. The aspect of comparability is illustrated in the simulation section of this paper.

In the same vein, we introduce the Variance of Anarchy (VoA) measure which can be defined as

$$\text{VoA} \triangleq \frac{\text{Var}_{\mathscr{P}_{J_T}}\{J_T(u^*)\}}{\sum_{i=1}^{N} \text{Var}_{\mathscr{P}_{J_i}}\{J_i(u_i^*)\}}. \tag{25}$$

The simulation results will examine the PoA and VoA for several classes of cost cumulant controllers.

VI. Decentralized Efficiency Loss Control Design

Consider now the following optimization for the noncooperative problem.

Definition 6.1 *(PoA-VoA Control Design):* Given the framework of the noncooperative and team (social welfare) optimal control problems, solve the optimization

$$\begin{aligned} &\min_{\theta\in[0,1]} \left\{ |\text{PoA}(\theta)-1| + |\text{VoA}(\theta)-1| \right\} \\ &\text{subject to: eq. of motion (8), (10), (11),} \\ &\qquad\qquad \text{eq. of motion (18), (20), (21)} \end{aligned} \tag{26}$$

in order to find θ^* that characterizes optimal cost density-shaping controls $k_1^*(\alpha,\theta^*)$, $k_2^*(\alpha,\theta^*)$ as given in (13), (23).

Choosing the controls per the optimization (26) will yield noncooperative control solutions for Players 1 and 2 that minimize efficiency loss due to noncooperation while maintaining a high level of confidence in the PoA metric (e.g. VoA near unity).

VII. Simulation Results

So far in this work, we have established two problems, one involving a team and another involving two competing agents. The objective is to compute the PoA and VoA for a family of cost densities approximately achieved by cost density-shaping controllers aimed to track targets (9) and (19), and thereby solve the optimization (26). These controllers will be computed via the Statistical Target Selection (STS) procedure described in Chapter 8 of [3]. This section will present the results of the controller computations accomplished with STS, and the optimal solution.

It should be noted that if $u_1(t) = u_2(t) = u(t)$ in (5), then $J_1(u)+J_2(u) = J_T(u)$. It is in this sense that the costs for the noncooperative and team cases make for a fair comparison of PoA and VoA.

It was determined from simulations that there is nearly a 32% increase in PoA and a 7.5% increase in PoA for $\mu = 1$, so this value will be chosen to characterize the $2CC$ control and carry out the STS procedure that iteratively solves (12) and (22) for $\theta \in [0,1]$. The detailed results behind the selection of that particular value have been excluded due to space limitations.

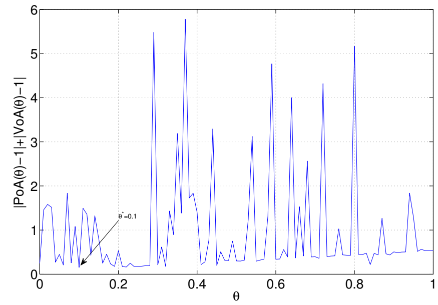

Fig. 2: PoA-VoA Control Design

Using $\mu = 1$, the STS design can be completed using the same tolerance and time step that was chosen when determining μ. Again, we reiterate that the objective here is to solve (12) and (22) for $\theta \in [0,1]$. For simplicity in the computation, we choose $\Delta\theta = 0.01$ to characterize the parametric target set. In Figure 2 it can be seen that there are mixing parameters $\theta \in [0,1]$ for which the normalized error is significant on the time-horizon $[0,1 \text{ sec}]$. This error is computed with the initial cumulants, and is shown below

$$\begin{aligned} err(\kappa_i^x) &= \frac{|\kappa_i^x(t_0) - \tilde{\kappa}_i^x(t_0,\theta)|}{\tilde{\kappa}_i^x(t_0,\theta)} \\ i &= 1,2,\ x = 1,2 \text{ (Player no.) T (Team)} \end{aligned} \tag{27}$$

The normalized error (27) is large for some θ in both the team and noncooperative cases. This simply means that the target could not be achieved for a certain mixing parameter. For PoA and VoA, the case of large error does manifest itself in the computation of both metrics. When error is not coincident for the noncooperative and team cost function cumulants, the PoA and VoA become much larger than one. In the case of coincident error, error magnitudes differ appreciably. The way the objective function has been defined for (26) makes the corresponding parameter values that give large error non-optimal. The objective function is shown over the parameter space in Figure 2.

We find that $\theta^* = 0.1$ is optimal over $\theta \in [0,1]$ per the criterion (26). The value of the objective function achieved with $\theta = 1$ ($2CC$) is 347% larger than that achieved with θ^*. What this means is that simulation of the system states $x(t)$ and $x^T(t)$ under the control laws corresponding to (θ^*, θ_{2CC}) should reveal large differences between the states for $2CC$ control, and smaller differences between the states for the optimal cost density-shaping control for θ^*. Recall that $x(t)$ and $x^T(t)$ are the queue lengths for the team and noncooperative models, respectively.

From from Figure 3, it can be seen qualitatively that $2CC$ control has more efficiency loss due to noncooperation than does the control corresponding to θ^*. The system state trajectories for team versus noncooperative cases are significantly different during some subintervals of $[0,100 \text{ sec}]$

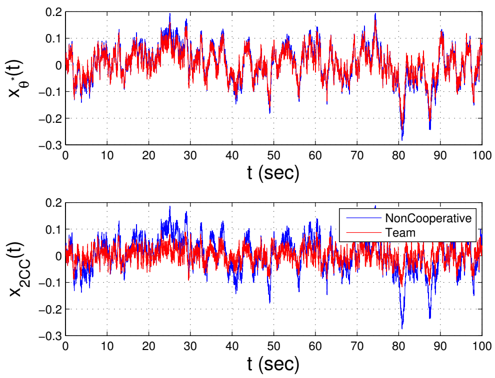

Fig. 3: $x(t)$ and $x_T(t)$ for (θ^*, θ_{2CC})

under $2CC$ control. We use the MATLAB function $std(\cdot)$ to get a quantitative comparison of a RMS measurement on the differences between $x(t)$ and $x_T(t)$ for each control solution. We found that σ_{2CC} is approximately 302% of σ_{θ^*}, where

$$\sigma_{\theta^*} = std\left(x_{\theta^*}(t) - x^T_{\theta^*}(t)\right), \ \sigma_{2CC} = std\left(x_{2CC}(t) - x^T_{2CC}(t)\right).$$

VIII. Summary and Conclusion

This paper introduces the VoA measure to quantify a level of confidence in PoA metrics for efficiency loss in dynamic games due to decentralized mechanisms. Furthermore, this work proposes a new design method for decentralized control algorithms that involves both the PoA and VoA. The new design method is illustrated using two separate optimal control problems - the noncooperative 2-Player problem and the team problem - where the underlying system is a queuing dynamics server. We identify a $2CC$ statistical controller that increases PoA beyond LQG control by over 32%. The cost cumulants for the team and noncooperative costs under LQG and this $2CC$ control are used to characterize a parametric family of target cost cumulants. The STS design procedure is used to formulate cost density-shaping controllers that seek to achieve each target in the family, and enables the PoA-VoA control design optimization to be solved over a finite parameter space. Simulation results support that the PoA-VoA optimization solution is a noncooperative control solution with higher efficiency than the case when both Players use $2CC$ controllers.

The initial findings in this work are interesting and warrant continued investigation of the PoA-VoA optimal control design methodology. While the application studied herein pertains only to 2 Players for the noncooperative case, there is nothing to indicate that the same VoA-PoA optimization could not be solved for N players. Additionally, the VoA is introduced here as a useful measure to add information (i.e. its variation) about the PoA metric, but this introduction does not preclude other cumulant-based measures of efficiency loss from being of practical importance. For instance, future work might study the inclusion of higher-order cumulants, such as skew or kurtosis, in efficiency metrics.

This work represents a new direction for applications of statistical control theory and existing efficiency measures for problems fitting a game theoretic framework and is likely to encourage research in the field.

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