Secure and Resilient Distributed Machine Learning Algorithms Under Adversarial Environment

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Motivations

Machine learning algorithms are widely used.

They are vulnerable to attackers who can create "misleading" data.

Distributed algorithms are developed to solve large-scale problems.

However, the system becomes more vulnerable.

- Enlarge attack surface.
- Network effects.

Objective of the talk

We study secure and resilient distributed machine learning algorithms under adversarial environment.

Related Works

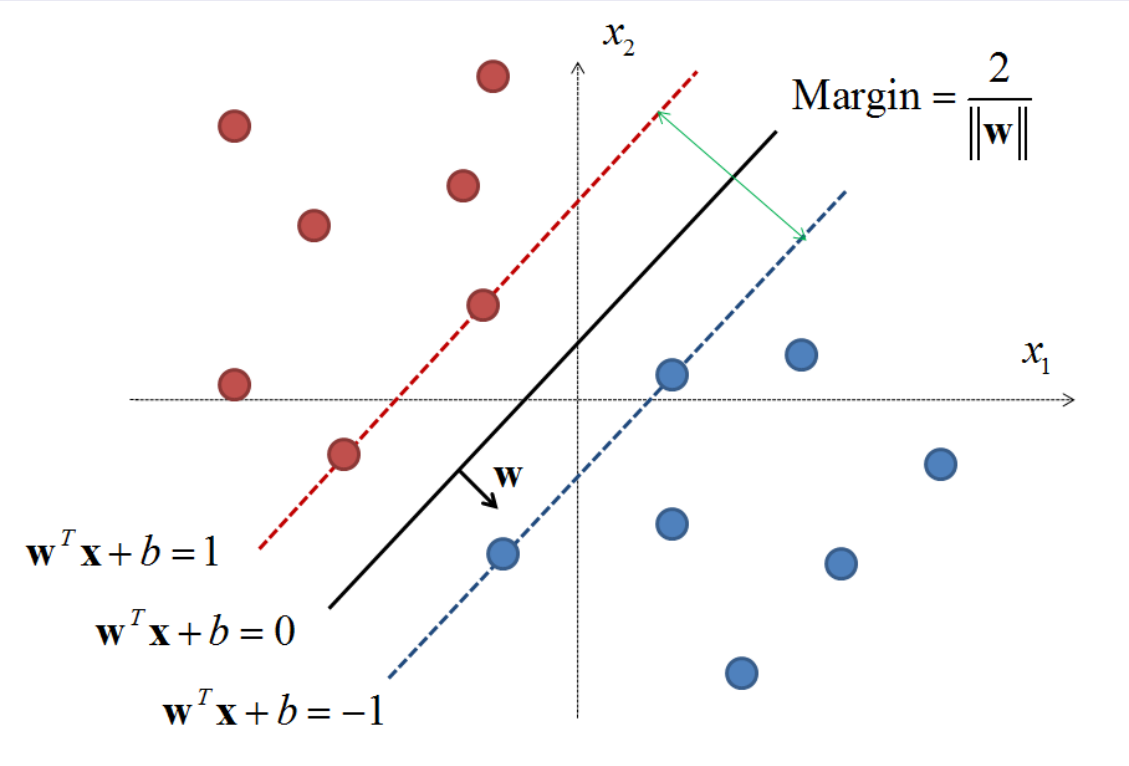
- Nilesh Dalvi, Pedro Domingos, Sumit Sanghai, and Deepak Verma. Adversarial classification. *Proceedings of the tenth ACM SIGKDD international conference on Knowledge discovery and data mining,* pp. 99-108. ACM, 2004.

- Wei Liu, and Sanjay Chawla. A game theoretical model for adversarial learning. *Proceedings of IEEE International Conference on Data Mining Workshops,* pp. 25-30. IEEE, 2009.

- Marco Barreno, Blaine Nelson, Anthony Joseph, and Doug Tygar. The security of machine learning. *Machine Learning* pp. 121-148. 2010.

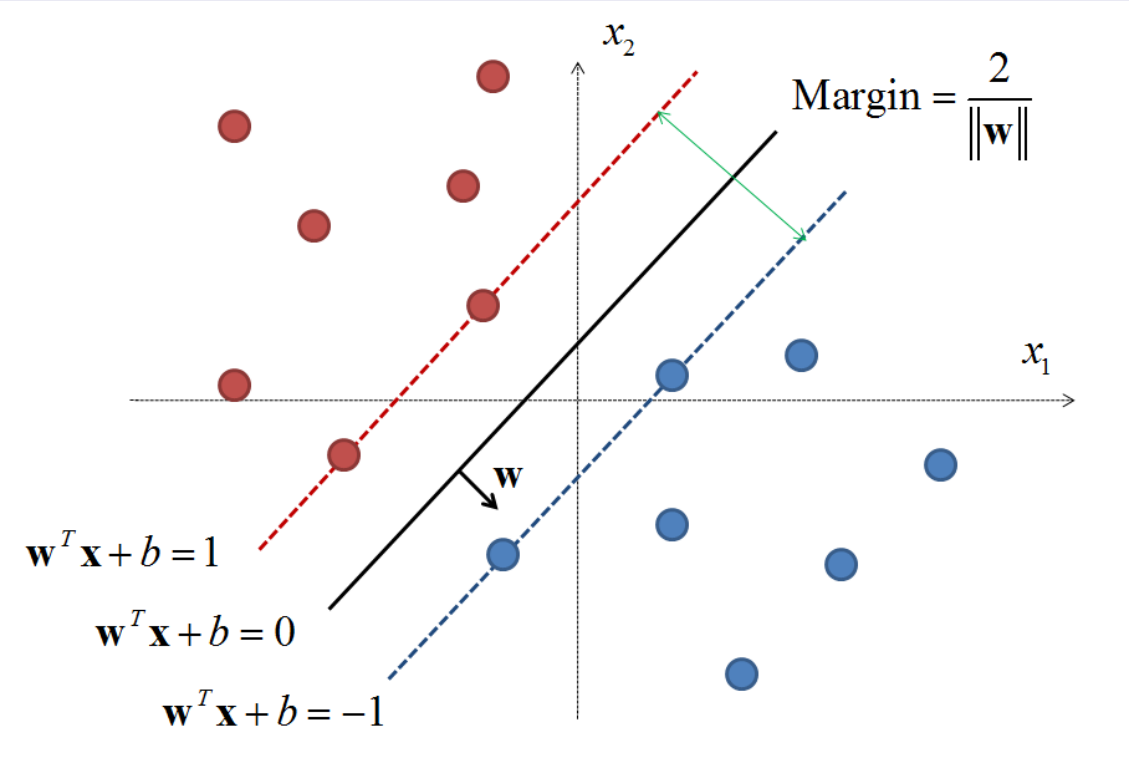
Distributed Support Vector Machines

Introduction to Support Vector Machines (SVMs)



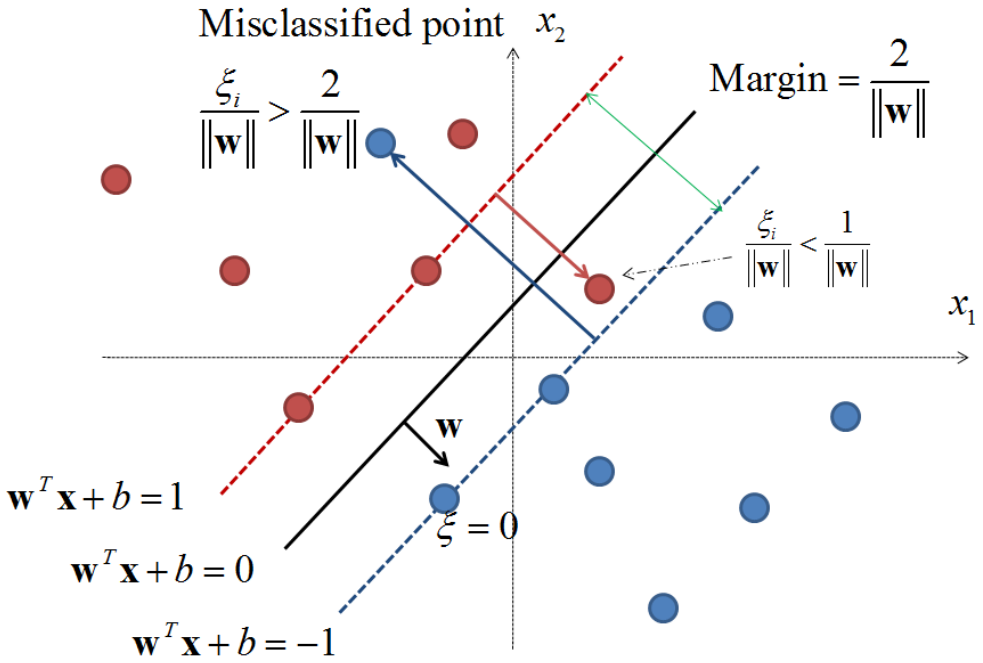
- SVMs are supervised learning models used for classification.
- The goal is to find the hyperplane with largest margins between different sets of samples [Cortes and Vapnik, 1995].

Introduction to Support Vector Machines (SVMs)



$$\min_{\mathbf{w}} \frac{1}{2}\|\mathbf{w}\|^2$$
$$\text{s.t.} \quad y_i\left(\mathbf{w}^T\mathbf{x}_i + b\right) \geq 1, \quad i = 1, ..., N.$$

Distributed Support Vector Machines

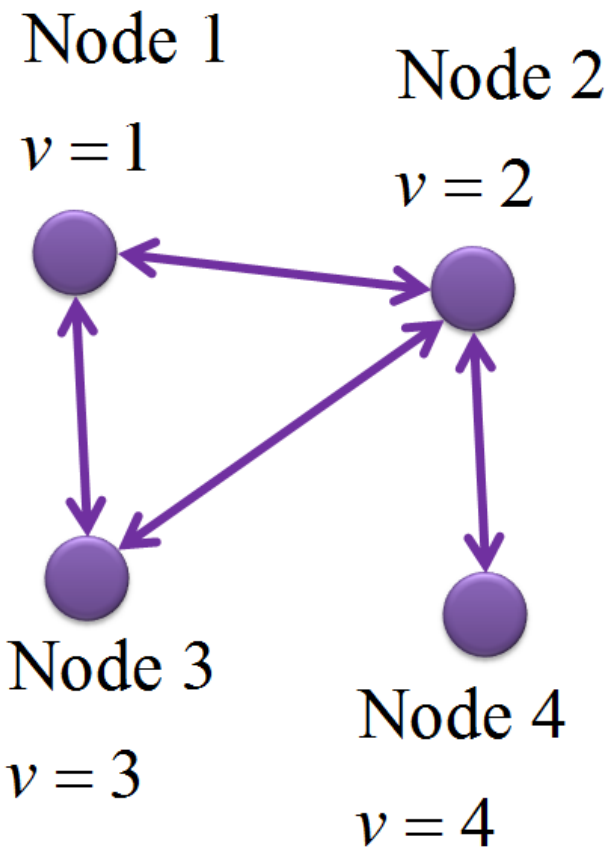


$$\min_{\mathbf{w},\{\xi_i\}} \ \frac{1}{2}\|\mathbf{w}\|^2 + C\sum_{i=1}^{N}\xi_i$$

$$\text{s.t.} \quad y_i\left(\mathbf{w}^T\mathbf{x}_i + b\right) \geq 1-\xi_i, \qquad i=1,...,N;$$

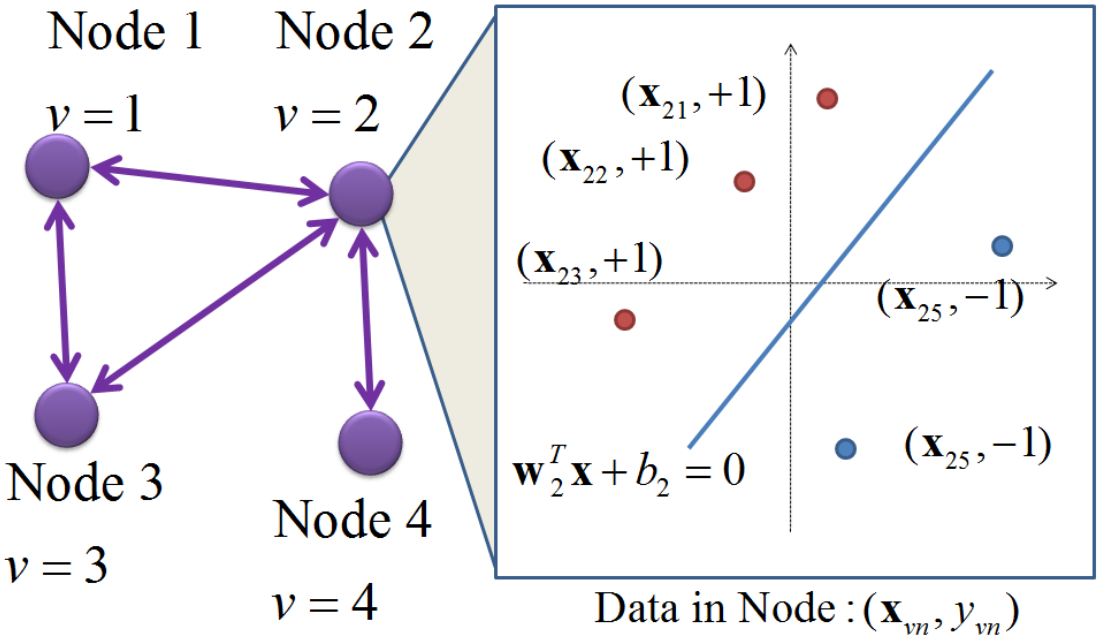
$$\xi_i \geq 0, \qquad i=1,...,N.$$

Distributed Support Vector Machines



- A network consists of 4 nodes.
- Nodes can communicate with their neighbors.
- Nodes do not have to be fully connected.

Distributed Support Vector Machines

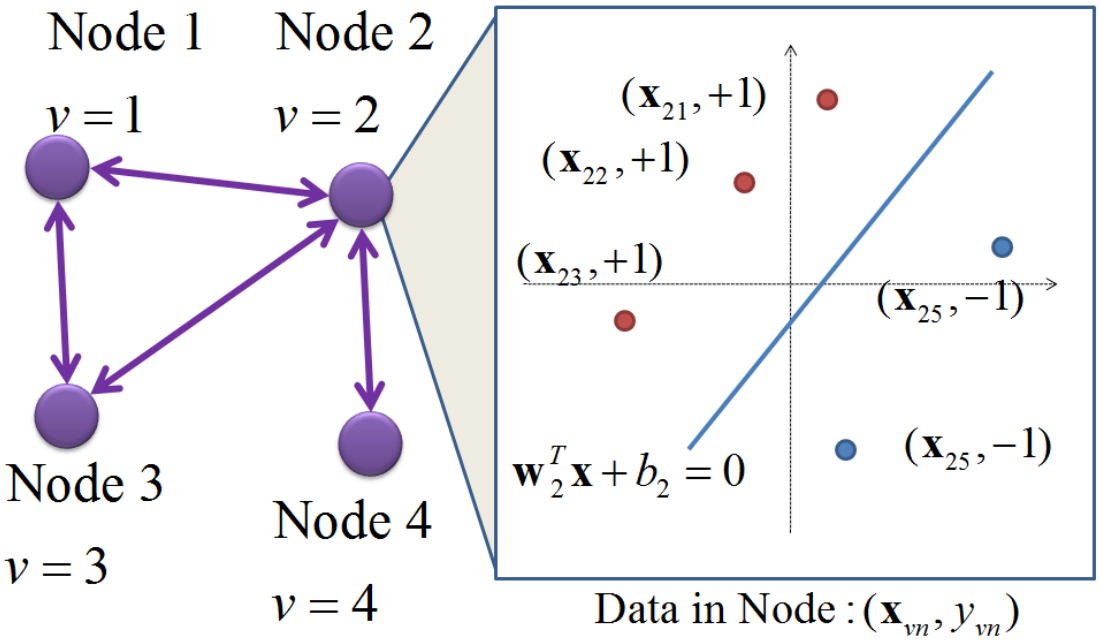


- Each node contains a labeled training set

$$\mathcal{D}_v := \{(\mathbf{x}_{vn}, y_{vn}) : n = 1, ..., N_v\}.$$

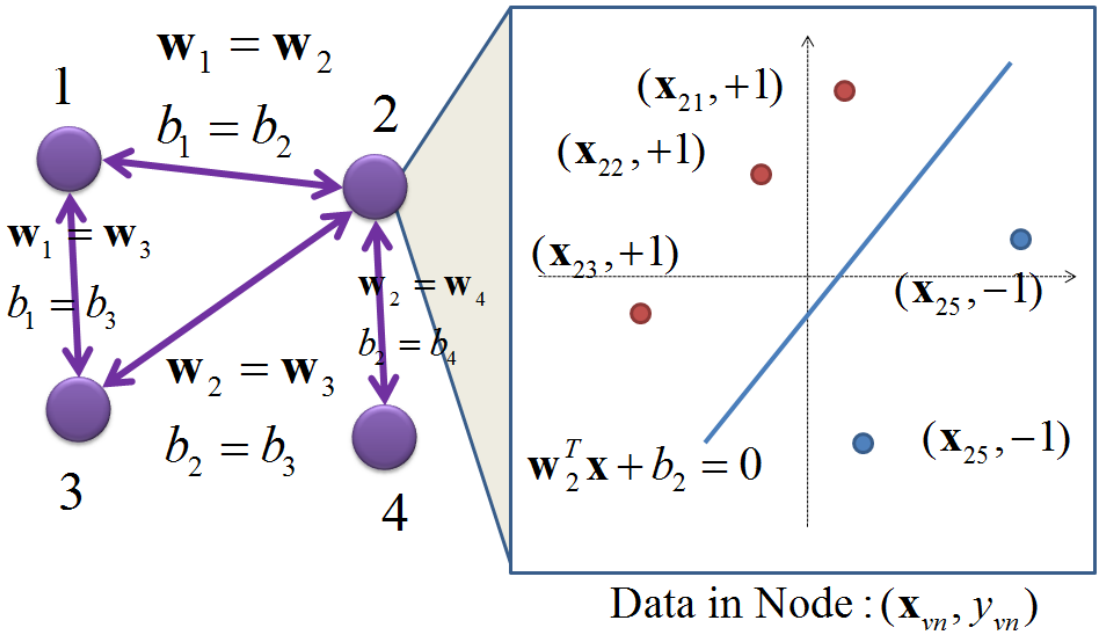
- $\mathbf{x}_{vn}$ is the n-th data in node v.
- y_{vn} is the label of n-th data in node v.

Distributed Support Vector Machines



- Pedro A. Forero, Alfonso Cano, and Georgios B. Giannakis. Consensus-based distributed support vector machines. *The Journal of Machine Learning Research* 11 (2010): 1663-1707.

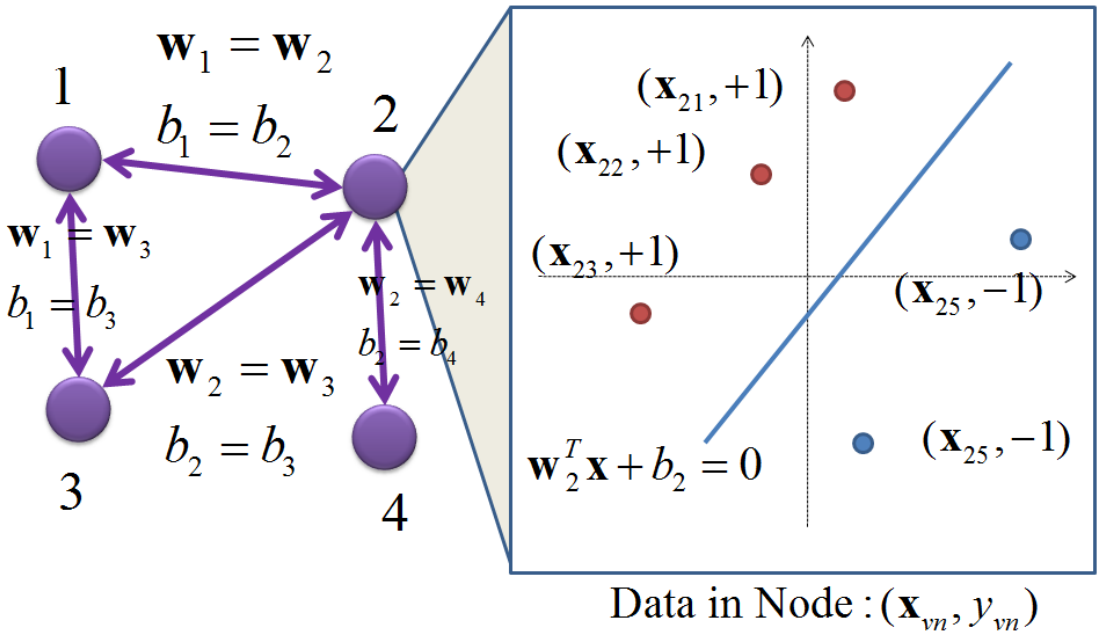
Distributed Support Vector Machines



$$\min_{\{\mathbf{w}_v, b_v, \{\xi_{vn}\}\}} \frac{1}{2} \sum_{v=1}^{V} \|\mathbf{w}_v\|^2 + V C_l \sum_{v=1}^{V} \sum_{n=1}^{N_v} \xi_{vn}$$

$$\text{s.t.} \quad \begin{array}{ll} y_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v) \geq 1 - \xi_{vn}, & \forall v \in \mathcal{V}, n = 1, ..., N_v; \\ \xi_{vn} \geq 0, & \forall v \in \mathcal{V}, n = 1, ..., N_v; \\ \mathbf{w}_v = \mathbf{w}_u, b_v = b_u, & \forall v \in \mathcal{V}, u \in \mathcal{B}_v. \end{array} \tag{1}$$

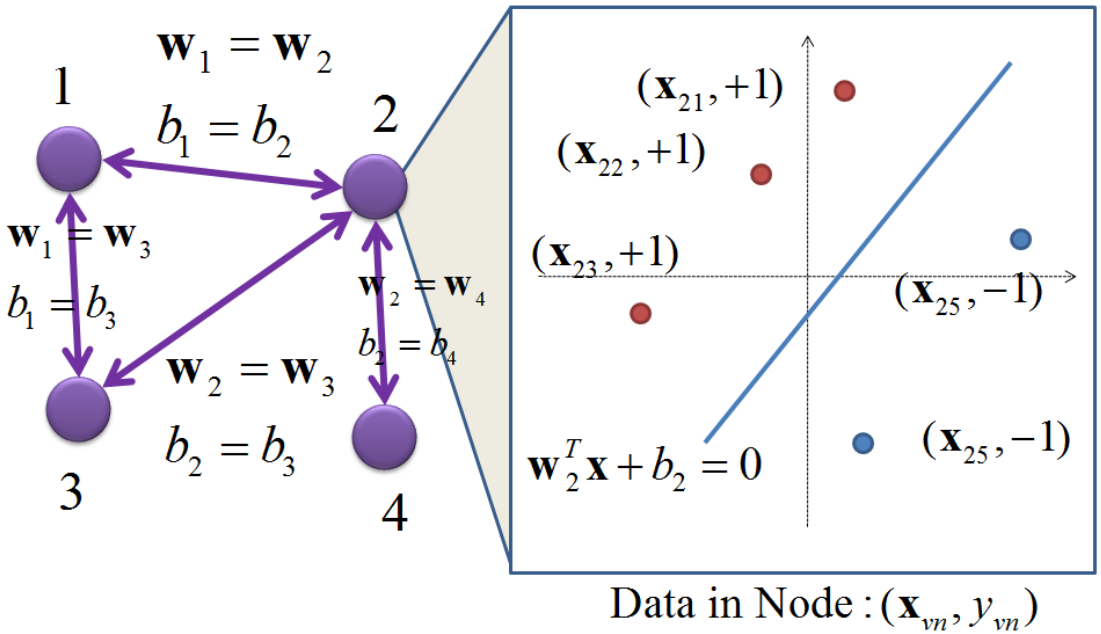
Distributed Support Vector Machines



$$\min_{\{\mathbf{w}_v, b_v, \{\xi_{vn}\}\}} \frac{1}{2} \sum_{v=1}^{V} \|\mathbf{w}_v\|^2 + V C_l \sum_{v=1}^{V} \sum_{n=1}^{N_v} \xi_{vn}$$

$$\begin{aligned} & y_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v) \geq 1 - \xi_{vn}, && \forall v \in \mathcal{V}, n = 1, ..., N_v; \\ \text{s.t.} \quad & \xi_{vn} \geq 0, && \forall v \in \mathcal{V}, n = 1, ..., N_v; \\ & \mathbf{w}_v = \mathbf{w}_u, b_v = b_u, && \forall v \in \mathcal{V}, u \in \mathcal{B}_v. \end{aligned}$$

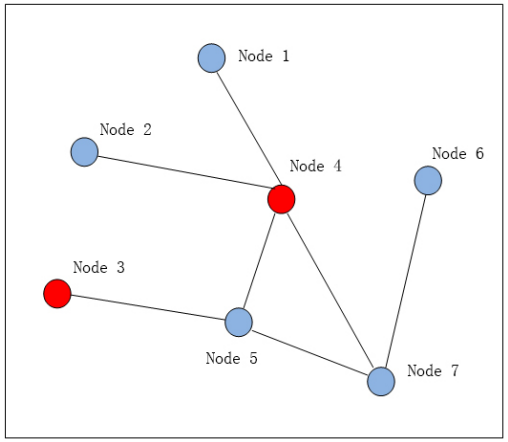
Distributed Support Vector Machines



$$\min_{\{\mathbf{w}_v, b_v\}} \frac{1}{2} \sum_{v=1}^{V} \|\mathbf{w}_v\|^2 + V C_l \sum_{v=1}^{V} \sum_{n=1}^{N_v} \left[1 - y_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+ \tag{2}$$

$$\text{s.t.} \quad \mathbf{w}_v = \mathbf{w}_u, \quad b_v = b_u, \quad \forall v \in \mathcal{V}, u \in B_v.$$

Problem Statement: Overview



- Each node contains a labeled training set $\mathcal{D}_v := \{(\mathbf{x}_{vn}, y_{vn}) : n = 1, ..., N_v\}$.
- Node 4 can communicate with its 4 neighbors. An attacker can take over a subset of the nodes.

System Model

- Aim of Learner: High Accuracy
- Aim of Attacker: Low Accuracy

Attack Model

- Attacker knows learning algorithm.
- Attacker can take over a set of nodes.
- Attacker can modify the training data.

A game between a learner and an attacker

Problem Statement: Mathematical Model

Mathematical Model for Learner

$$\min_{\{\mathbf{w}_v, b_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)} + \underbrace{V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \widehat{\mathbf{x}}_{vn} + b_v)\right]_+}_{(c)} \tag{3}$$

$$\text{s.t.} \quad \mathbf{w}_v = \mathbf{w}_u, \quad b_v = b_u, \quad \forall v \in \mathcal{V}, u \in \mathcal{B}_v.$$

- Term (a) is the sum of inverse of the distance of margin.
- Term (b) is the error penalty of nodes without attack.
- Term (c) is the error penalty of nodes under attack.
- $\widehat{\mathbf{x}}_{vn}$ represents data modified by the attacker.

Problem Statement: Mathematical Model

Mathematical Model for Learner

$$\min_{\{\mathbf{w}_v, b_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1 - y_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$
$$+ \underbrace{V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1 - y_{vn}(\mathbf{w}_v^T \widehat{\mathbf{x}}_{vn} + b_v)\right]_+}_{(c)}$$
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Problem Statement: Mathematical Model

Mathematical Model for Learner

$$\min_{\{\mathbf{w}_v, b_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_I C_I \sum_{v=1}^{V_I}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$
$$+ \underbrace{V_a C_I \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \widehat{\mathbf{x}}_{vn} + b_v)\right]_+}_{(c)}$$
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Problem Statement: Mathematical Model

Mathematical Model for Learner

$$\min_{\{\mathbf{w}_v, b_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$
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Problem Statement: Mathematical Model

Mathematical Model for Learner

$$\min_{\{\mathbf{w}_v, b_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$
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Problem Statement: Mathematical Model

Mathematical Model for the Attacker

- The attacker changes $\mathbf{x}_{vn}$ into $\widehat{\mathbf{x}}_{vn} = \mathbf{x}_{vn} - \delta_{vn}$, where $\{\delta_{vn}\} \in \mathcal{U}$, and $\mathcal{U}$ is the attacker's action set [Xu et al., 2009].
- Here

$$\mathcal{U} = \left\{ (\delta_{v1}, \delta_{v2}, ..., \delta_{vN_v}) \left| \sum_{n=1}^{N_v} \|\delta_{vn}\| \leq C_\delta \right. \right\}$$

- Also we introduce

$$\mathcal{U}_0 = \{\delta_v \,|\|\delta_v\| \leq C_\delta\}$$

- Notice that C_δ does not have to be same in different nodes.

Problem Statement: Mathematical Model

Mathematical Model for Attacker

$$\max_{\{\delta_{vn}\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1-\mathrm{y}_{vn}(\mathbf{w}_v^T\mathbf{x}_{vn}+b_v)\right]_+}_{(b)}$$
$$+ \underbrace{V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1-\mathrm{y}_{vn}(\mathbf{w}_v^T(\mathbf{x}_{vn}-\delta_{vn})+b_v)\right]_+}_{(c)} \underbrace{- C_a \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\|\delta_{vn}\|_0}_{(d)}$$
$$\text{s.t. } (\delta_{v1},...,\delta_{vN_v}) \in \mathcal{U}_v, \quad \forall v \in \mathcal{V}_a. \tag{4}$$

- Terms (a)(b)(c) are the objectives of the learner's min-problem.
- Term (d) is the cost function for the attacker.
- l_0 norm is a total number of nonzero elements in a vector.

Problem Statement: Mathematical Model

Mathematical Model for Attacker

$$\max_{\{\delta_{vn}\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1-\mathrm{y}_{vn}(\mathbf{w}_v^T\mathbf{x}_{vn}+b_v)\right]_+}_{(b)}$$
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$$\underbrace{+ V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1-\mathrm{y}_{vn}(\mathbf{w}_v^T(\mathbf{x}_{vn}-\delta_{vn})+b_v)\right]_+}_{(c)} \underbrace{- C_a \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\|\delta_{vn}\|_0}_{(d)}$$

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Mathematical Model:Min-Max Problem

$$\min_{\{\mathbf{w}_v, b_v\}} \max_{\{\delta_{vn}\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$

$$\underbrace{+ V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T(\mathbf{x}_{vn} - \delta_{vn}) + b_v)\right]_+}_{(c)} \underbrace{- C_a \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\|\delta_{vn}\|_0}_{(d)}$$

$$\text{s.t.} \quad \begin{array}{ll} \mathbf{w}_v = \mathbf{w}_u, b_v = b_u, & \forall v \in \mathcal{V}, u \in \mathcal{B}_v; \\ (\delta_{v1}, ..., \delta_{vN_v}) \in \mathcal{U}_v, & \forall v \in \mathcal{V}_a. \end{array} \tag{5}$$

- Terms $(a)(b)(c)$ are related to the min-problem for the learner.
- Terms $(a)(b)(c)(d)$ are related to the max-problem for the attacker.
- A nonzero-sum game.

Mathematical Model:Min-Max Problem

$$\min_{\{\mathbf{w}_v, b_v\}} \max_{\{\delta_{vn}\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V_l C_l \sum_{v=1}^{V_l}\sum_{n=1}^{N_v}\left[1 - y_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$

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$$+ \underbrace{V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1 - \mathrm{y}_{vn}(\mathbf{w}_v^T(\mathbf{x}_{vn} - \delta_{vn}) + b_v)\right]_+}_{(c)} \underbrace{- C_a \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\|\delta_{vn}\|_0}_{(d)}$$

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$$+ \underbrace{V_a C_l \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\left[1 - y_{vn}(\mathbf{w}_v^T(\mathbf{x}_{vn} - \delta_{vn}) + b_v)\right]_+}_{(c)} - \underbrace{C_a \sum_{v=1}^{V_a}\sum_{n=1}^{N_v}\|\delta_{vn}\|_0}_{(d)}$$

$$\text{s.t.} \quad \begin{array}{ll} \mathbf{w}_v = \mathbf{w}_u, b_v = b_u, & \forall v \in \mathcal{V}, u \in \mathcal{B}_v; \\ (\delta_{v1}, ..., \delta_{vN_v}) \in \mathcal{U}_v, & \forall v \in \mathcal{V}_a. \end{array}$$

- Terms $(c)(d)$ can be simplified further by using the definition of $\mathcal{U}_v$ and $\mathcal{U}_{v0}$.

Solutions and Algorithms:Proposition 1

With $\mathcal{U}_v = \left\{ (\delta_{v1}, ..., \delta_{vN_v}) \,\middle|\, \sum_{n=1}^{N_v} \|\delta_{vn}\| \leq C_\delta \right\}$ and $\mathcal{U}_{v0} = \{\delta_v \,|\, \|\delta_v\| \leq C_\delta\}$, the min-max problem is equivalent to the following problem:

$$\min_{\{\mathbf{w}_v, b_v\}} \max_{\{\delta_{vn}\}} \underbrace{\frac{1}{2} \sum_{v=1}^{V} \|\mathbf{w}_v\|^2}_{(a)} + \underbrace{V C_l \sum_{v=1}^{V} \sum_{n=1}^{N_v} \left[1 - y_{vn}(\mathbf{w}_v^T \mathbf{x}_{vn} + b_v)\right]_+}_{(b)}$$
$$+ \underbrace{V_a C_l \sum_{v=1}^{V_a} \mathbf{w}_v^T \delta_v}_{(c)} - \underbrace{C_a \sum_{v=1}^{V_a} \|\delta_v\|_0}_{(d)}, \tag{6}$$
$$\text{s.t.} \quad \begin{array}{ll} \mathbf{w}_v = \mathbf{w}_u, b_v = b_u, & \forall v \in \mathcal{V}, u \in \mathcal{B}_v; \\ \delta_v \in \mathcal{U}_{v0}, & \forall v \in \mathcal{V}_a. \end{array}$$

Solutions and Algorithms:Problem Reformulation

- Decision variables space $\mathbf{r}_v := [\mathbf{w}_v^T, b_v]^T$,
- Augmented matrix $\mathbf{X}_v := [(\mathbf{x}_{v1}, ..., \mathbf{x}_{vN_v})^T, \mathbf{1}_v]$,
- Diagonal label matrix $\mathbf{Y}_v := diag([y_{v1}, ..., y_{vN_v}])$.
- Note that $\mathbf{w}_v = (\mathbf{I}_{p+1} - \Pi_{p+1})\mathbf{r}_v$, where $\mathbf{I}_{p+1} - \Pi_{p+1}$ is a $(p+1) \times (p+1)$ identity matrix with 0 at $(p+1, p+1)$-st entry.

Thus, problem (6) can be rewritten as

$$\begin{aligned} \min_{\{\mathbf{r}_v, \omega_{vu}\}} \max_{\{\delta_v\}} & \frac{1}{2} \sum_{v=1}^{V} \mathbf{r}_v^T (\mathbf{I}_{p+1} - \Pi_{p+1}) \mathbf{r}_v + V C_l \sum_{v=1}^{V} [\mathbf{1}_v - \mathbf{Y}_v \mathbf{X}_v \mathbf{r}_v]_+ \\ & + V_a C_l \sum_{v=1}^{V_a} \mathbf{r}_v^T (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v - C_a \sum_{v=1}^{V_a} \|\delta_v\|_0 \qquad (7) \\ \text{s.t.} \quad & \mathbf{r}_v = \omega_{vu}, \omega_{vu} = \mathbf{r}_u, \qquad \forall v \in \mathcal{V}, \forall u \in \mathcal{B}_v; \\ & \delta_v \in \mathcal{U}_{v0}, \qquad \forall v \in \mathcal{V}_a. \end{aligned}$$

Solutions and Algorithms: Problem Reformulation

$$\min_{\{\mathbf{r}_v,\omega_{vu}\}} \max_{\{\delta_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\mathbf{r}_v^T(\mathbf{I}_{p+1}-\Pi_{p+1})\mathbf{r}_v}_{(a)} + \underbrace{VC_l\sum_{v=1}^{V}[\mathbf{1}_v-\mathbf{Y}_v\mathbf{X}_v\mathbf{r}_v]_+}_{(b)}$$
$$\underbrace{+V_aC_l\sum_{v=1}^{V_a}\mathbf{r}_v^T(\mathbf{I}_{p+1}-\Pi_{p+1})\delta_v}_{(c)} \underbrace{-C_a\sum_{v=1}^{V_a}\|\delta_v\|_0}_{(d)}$$
$$\text{s.t.} \quad \mathbf{r}_v=\omega_{vu},\omega_{vu}=\mathbf{r}_u, \quad \forall v\in\mathcal{V},\forall u\in\mathcal{B}_v;$$
$$\delta_v\in\mathcal{U}_{v0}, \quad \forall v\in\mathcal{V}_a.$$

- Terms $(c)(d)$ are related to the max-problem.
- Terms $(a)(b)(c)$ are related to the min-problem.
- A zero-sum game.

Solutions and Algorithms: Problem Reformulation

$$\min_{\{\mathbf{r}_v,\omega_{vu}\}} \max_{\{\delta_v\}} \underbrace{\frac{1}{2}\sum_{v=1}^{V}\mathbf{r}_v^T(\mathbf{I}_{p+1}-\Pi_{p+1})\mathbf{r}_v}_{(a)} + \underbrace{VC_l\sum_{v=1}^{V}[\mathbf{1}_v-\mathbf{Y}_v\mathbf{X}_v\mathbf{r}_v]_+}_{(b)}$$
$$\underbrace{+V_aC_l\sum_{v=1}^{V_a}\mathbf{r}_v^T(\mathbf{I}_{p+1}-\Pi_{p+1})\delta_v}_{(c)} - \underbrace{C_a\sum_{v=1}^{V_a}\|\delta_v\|_0}_{(d)}$$

$$\text{s.t.} \quad \begin{array}{ll} \mathbf{r}_v=\omega_{vu}, \omega_{vu}=\mathbf{r}_u, & \forall v\in\mathcal{V}, \forall u\in\mathcal{B}_v; \\ \delta_v\in\mathcal{U}_{v0}, & \forall v\in\mathcal{V}_a. \end{array}$$

- Terms $(c)(d)$ are related to the max-problem.
- Terms $(a)(b)(c)$ are related to the min-problem.
- A zero-sum game.

Construct Min-Problem and Max-Problem

- To solve the min-max problem, we use best response dynamics to construct the best response for the min-problem and max-problem separately.
- For fixed $\{\mathbf{r}_v^*\}$, the max-problem,

$$\begin{aligned} \max_{\{\delta_v\}} & \sum_{v=1}^{V_a} \left(V_a C_l \mathbf{r}_{\mathbf{v}}^{*T} (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v - C_a \left\| \delta_v \right\|_0 \right) \\ \text{s.t.} \quad & \delta_v \in \mathcal{U}_{v0}, \quad \forall v \in \mathcal{V}_a. \end{aligned} \tag{8}$$

- For a fixed $\{\delta_v^*\}$, the min-problem,

$$\begin{aligned} \min_{\{\mathbf{r}_v, \omega_{vu}\}} & \frac{1}{2} \sum_{v=1}^{V} \mathbf{r}_v^T (\mathbf{I}_{p+1} - \Pi_{p+1}) \mathbf{r}_v + V C_l \sum_{v=1}^{V} \left[\mathbf{1}_v - \mathbf{Y}_v \mathbf{X}_v \mathbf{r}_v \right]_+ \\ & + \sum_{v=1}^{V_a} V_a C_l \mathbf{r}_v^T (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v^* \\ \text{s.t.} \quad & \mathbf{r}_v = \omega_{vu}, \omega_{vu} = \mathbf{r}_u, \qquad \forall v \in \mathcal{V}, \forall u \in \mathcal{B}_v. \end{aligned} \tag{9}$$

Solutions and Algorithms

The Max-Problem

For fixed $\{\mathbf{r}_v^*, \delta\}$, the max-problem,

$$\begin{aligned} \max_{\{\delta_v\}} \sum_{v=1}^{V_a} \left(V_a C_I \mathbf{r_v^*}^T (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v - C_a \left\| \delta_v \right\|_0 \right) \\ \text{s.t.} \quad \delta_v \in \mathcal{U}_{v0}, \quad \forall v \in \mathcal{V}_a. \end{aligned} \tag{10}$$

Solutions to the Max-Problem

We relax l_0 norm to l_1 norm to represent the cost function of the attacker. By writing the dual form of the l_1 norm, we arrive at

$$\begin{aligned} \max_{\{\delta_v, s_v\}} \; & V_a C_I \mathbf{r_v^*}^T (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v - \mathbf{1}^T s_v \\ & C_a \delta_v \leq s_v, \qquad \forall v \in \mathcal{V}_a; \\ \text{s.t.} \quad & C_a \delta_v \geq -s_v, \qquad \forall v \in \mathcal{V}_a; \\ & \delta_v \in \mathcal{U}_{v0}, \qquad \forall v \in \mathcal{V}_a. \end{aligned} \tag{11}$$

Solutions and Algorithms

The Max-Problem

For fixed $\{\mathbf{r}_v^*, \delta\}$, the max-problem,

$$\max_{\{\delta_v\}} \sum_{v=1}^{V_a} \left(V_a C_I \mathbf{r}_{\mathbf{v}}^{*T} (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v - C_a \|\delta_v\|_0 \right)$$
$$\text{s.t.} \quad \delta_v \in \mathcal{U}_{v0}, \quad \forall v \in \mathcal{V}_a.$$

Solutions to the Max-Problem

We relax l_0 norm to l_1 norm to represent the cost function of the attacker. By writing the dual form of the l_1 norm, we arrive at

$$\max_{\{\delta_v, s_v\}} \; V_a C_I \mathbf{r}_{\mathbf{v}}^{*T} (\mathbf{I}_{p+1} - \Pi_{p+1}) \delta_v - \mathbf{1}^T s_v$$
$$\begin{aligned} & C_a \delta_v \le s_v, && \forall v \in \mathcal{V}_a; \\ \text{s.t.} \quad & C_a \delta_v \ge -s_v, && \forall v \in \mathcal{V}_a; \\ & \delta_v \in \mathcal{U}_{v0}, && \forall v \in \mathcal{V}_a. \end{aligned}$$

Solutions and Algorithms

The Min-Problem

For fixed $\{\delta_v^*\}$, the min-problem,

$$\begin{aligned}\min_{\{\mathbf{r}_v,\xi_v,\omega_{vu}\}} \ & \frac{1}{2}\sum_{v=1}^{V}\mathbf{r}_v^T(\mathbf{I}_{p+1}-\Pi_{p+1})\mathbf{r}_v + VC_l\sum_{v=1}^{V}[\mathbf{1}_v-\mathbf{Y}_v\mathbf{X}_v\mathbf{r}_v]_+ \\ & + \sum_{v=1}^{V_a} V_a C_l \mathbf{r}_v^T(\mathbf{I}_{p+1}-\Pi_{p+1})\delta_v^* \\ \text{s.t.} \quad & \mathbf{r}_v=\omega_{vu}, \omega_{vu}=\mathbf{r}_u, \qquad \forall v\in\mathcal{V}, \forall u\in\mathcal{B}_u.\end{aligned} \tag{12}$$

Solution to the Min-Problem

We use the alternating direction method of multipliers (ADMoM).

ADMoM for the Min-Problem

ADMoM is a distributed optimization algorithm [Boyd et al.,2010].

$$\begin{aligned} &\min_{\{\mathbf{r}_v,\omega\}} \sum_{v=1}^{V} f_v(\mathbf{r}_v) \\ &\text{s.t.} \quad \mathbf{r}_v - \omega = 0, \quad v = 1, ..., V. \end{aligned} \tag{13}$$

where f_v are convex functions.

With α_v denotes the Lagrange multipliers, the ADMoM solves problem (13) by the update rules below:

$$\mathbf{r}_v(t+1) \in \arg\min_{\mathbf{r}_v \in \mathcal{P}_1} f_v(\mathbf{r}_v) + \alpha_v^T(\mathbf{r}_v - \omega) + \frac{\eta}{2}\|\mathbf{r}_v - \omega\|^2. \tag{14}$$

$$\omega(t+1) \in \arg\min_{\omega \in \mathcal{P}_2} \sum_{v=1}^{V} f_v(\mathbf{r}_v) + \alpha_v^T(\mathbf{r}_v - \omega) + \frac{\eta}{2}\|\mathbf{r}_v - \omega\|^2. \tag{15}$$

$$\alpha_v(t+1) = \alpha_v(t) + \eta(\mathbf{r}_v(t+1) - \omega(t+1)). \tag{16}$$

Solutions to the Min-Problem

- Apply ADMoM to the Min-Problem, and use iterative methods to find solutions.

Lemma Each node iterates $\lambda_v(t), \mathbf{r}_v(t)$ and $\alpha_v(t)$, given by

$$\begin{aligned}\lambda_v(t+1) \in \arg \max_{\mathbf{0} \le \lambda_v \le V C_l \mathbf{1}_v} & -\tfrac{1}{2}\lambda_v^T \mathbf{Y}_v \mathbf{X}_v \mathbf{U}_v^{-1} \mathbf{X}_v^T \mathbf{Y}_v \lambda_v \\ & + (\mathbf{1}_v + \mathbf{Y}_v \mathbf{X}_v \mathbf{U}_v^{-1} \mathbf{f}_v(t))^T \lambda_v,\end{aligned} \tag{17}$$

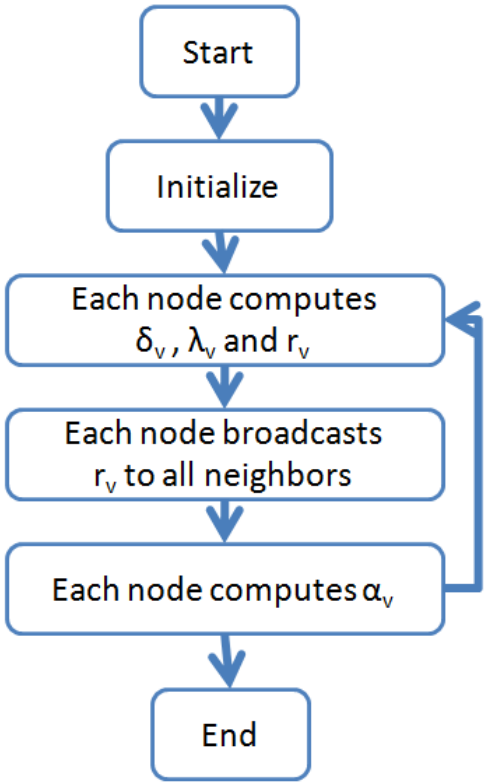
$$\mathbf{r}_v(t+1) = \mathbf{U}_v^{-1} \left(\mathbf{X}_v^T \mathbf{Y}_v \lambda_v(t+1) - \mathbf{f}_v(t) \right), \tag{18}$$

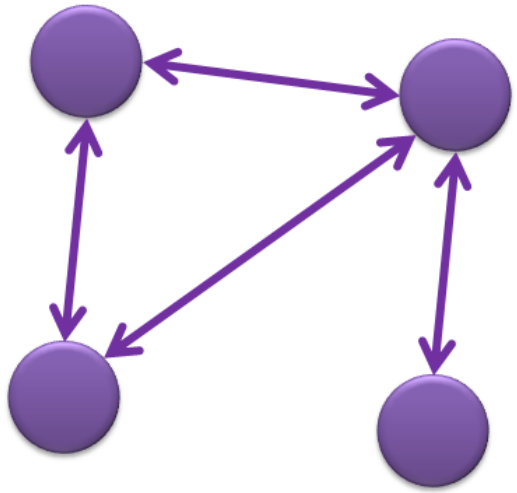
$$\alpha_v(t+1) = \alpha_v(t) + \frac{\eta}{2} \sum_{u \in \mathcal{B}_v} [\mathbf{r}_v(t+1) - \mathbf{r}_u(t+1)], \tag{19}$$

- $\mathbf{U}_v = (\mathbf{I}_{p+1} - \Pi_{p+1}) + 2\eta|\mathcal{B}_v|\mathbf{I}_{p+1}$
- $\mathbf{f}_v(t) = V_a C_l \delta_v^* + 2\alpha_v(t) - \eta \sum_{u \in \mathcal{B}_v} (\mathbf{r}_v(t) + \mathbf{r}_u(t)), \eta > 0.$

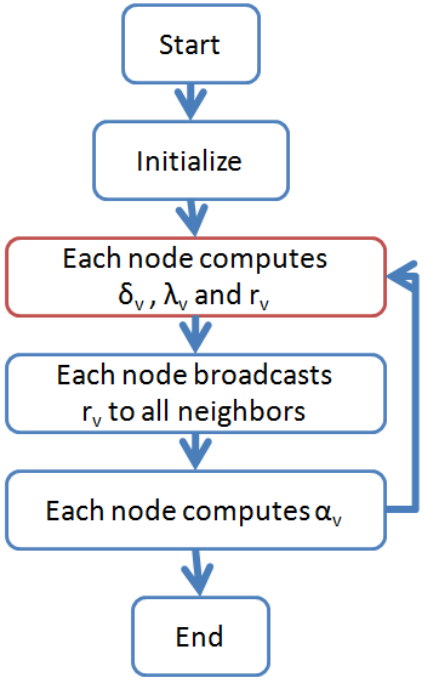
Secure and Resilient Algorithms

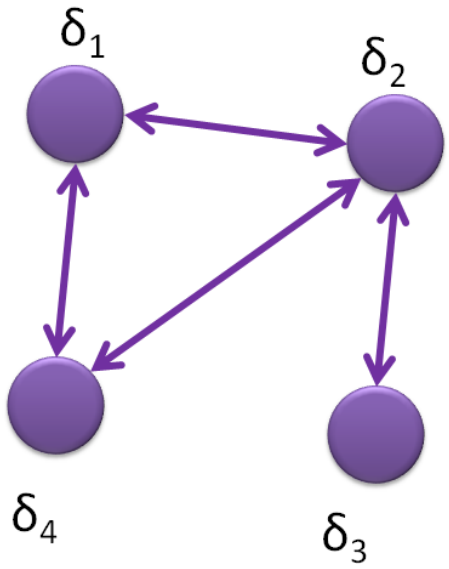
- Combine the solutions of Max-Problem with the solutions of Min-Problem.





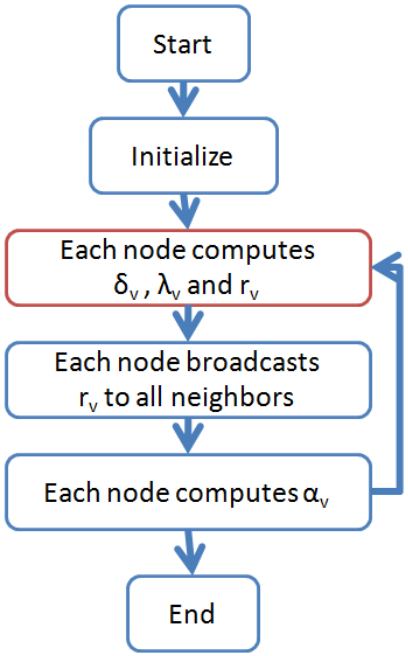
Algorithms:Step 1 Each Node Computes δ_v

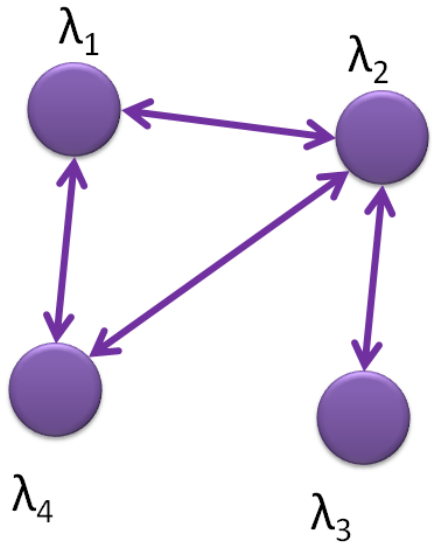




$$
\begin{aligned}
\delta_v(t+1) \in \arg \max_{\{\delta_v, s_v\}} \; & V_a C_I \mathbf{r}_v^T(t)(\mathbf{I}_{p+1} - \Pi_{p+1})\delta_v - \mathbf{1}^T s_v \\
& C_a \delta_v \leq s_v, \qquad \forall v \in \mathcal{V}_a; \\
\text{s.t.} \quad & C_a \delta_v \geq -s_v, \qquad \forall v \in \mathcal{V}_a; \\
& \delta_v \in \mathcal{U}_{v0}, \qquad \forall v \in \mathcal{V}_a.
\end{aligned} \tag{20}
$$

Algorithms:Step 2 Each Node Computes λ_v





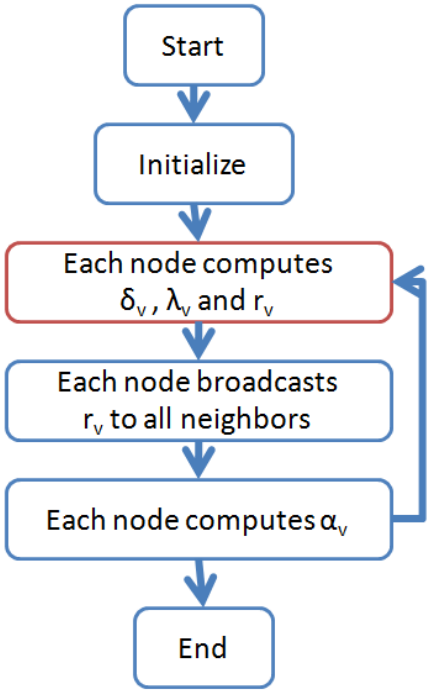
$$\lambda_v(t+1) \in \arg \max_{\mathbf{0} \le \lambda_v \le VC_l\mathbf{1}_v} -\frac{1}{2}\lambda_v^T \mathbf{Y}_v \mathbf{X}_v \mathbf{U}_v^{-1} \mathbf{X}_v^T \mathbf{Y}_v \lambda_v + \left(\mathbf{1}_v + \mathbf{Y}_v \mathbf{X}_v \mathbf{U}_v^{-1} \mathbf{f}_v(t)\right)^T \lambda_v, \tag{21}$$

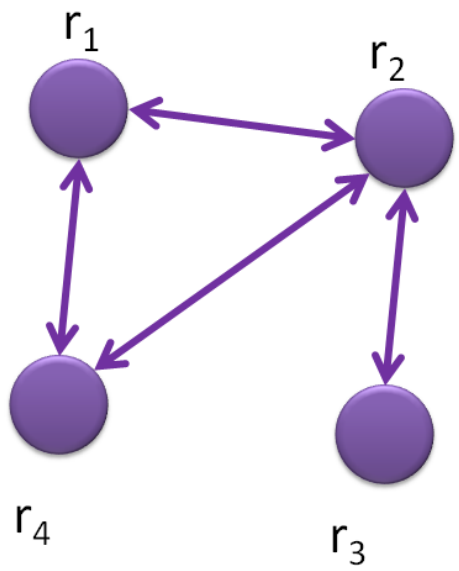
where

$$\mathbf{U}_v = (\mathbf{I}_{p+1} - \Pi_{p+1}) + 2\eta|\mathcal{B}_v|\mathbf{I}_{p+1},$$

$$\mathbf{f}_v(t) = V_a C_l \delta_v(t) + 2\alpha_v(t) - \eta \sum_{u \in \mathcal{U}_v} (\mathbf{r}_v(t) + \mathbf{r}_u(t)).$$

Algorithms:Step 3 Each Node Computes $\mathbf{r}_v$





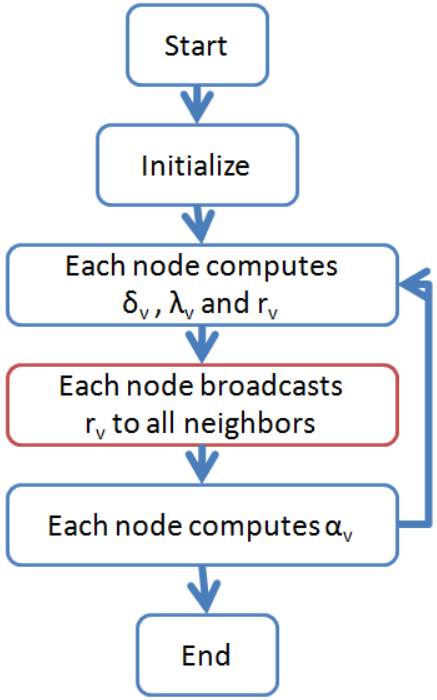
$$\mathbf{r}_v(t+1) = \mathbf{U}_v^{-1}\left(\mathbf{X}_v^T\mathbf{Y}_v\lambda_v(t+1) - \mathbf{f}_v(t)\right), \tag{22}$$

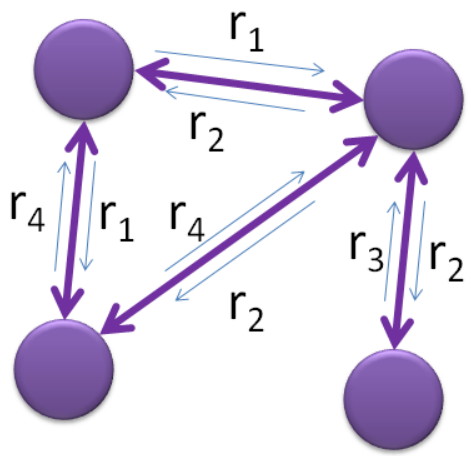
where

$$\mathbf{U}_v = (\mathbf{I}_{p+1} - \Pi_{p+1}) + 2\eta|\mathcal{B}_v|\mathbf{I}_{p+1},$$

$$\mathbf{f}_v(t) = V_a C_l \delta_v(t) + 2\alpha_v(t) - \eta \sum_{u\in\mathcal{U}_v}(\mathbf{r}_v(t) + \mathbf{r}_u(t)).$$

Algorithms:Step 4 Each Node Broadcasts $\mathbf{r}_v$ to Neighbors





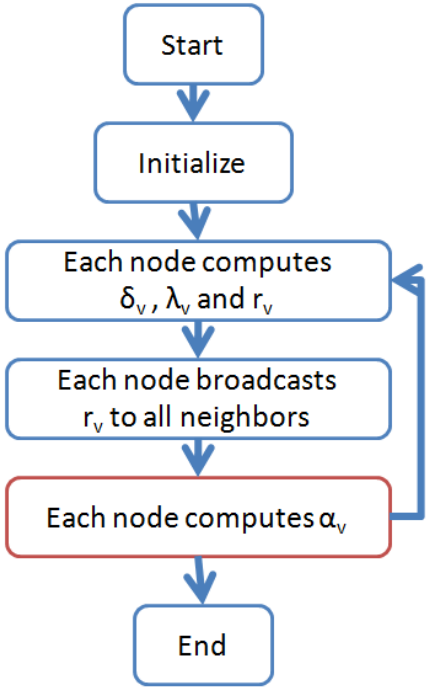
$$\mathbf{r}_v(t+1) = \mathbf{U}_v^{-1}\left(\mathbf{X}_v^T\mathbf{Y}_v\lambda_v(t+1) - \mathbf{f}_v(t)\right), \tag{23}$$

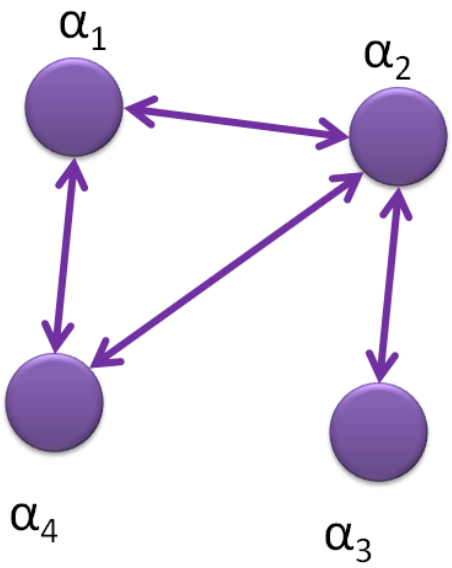
where

$$\mathbf{U}_v = (\mathbf{I}_{p+1} - \Pi_{p+1}) + 2\eta|\mathcal{B}_v|\mathbf{I}_{p+1},$$

$$\mathbf{f}_v(t) = V_a C_l \delta_v(t) + 2\alpha_v(t) - \eta \sum_{u\in\mathcal{U}_v}(\mathbf{r}_v(t) + \mathbf{r}_u(t)).$$

Algorithms:Step 5 Each Node Computes α_v

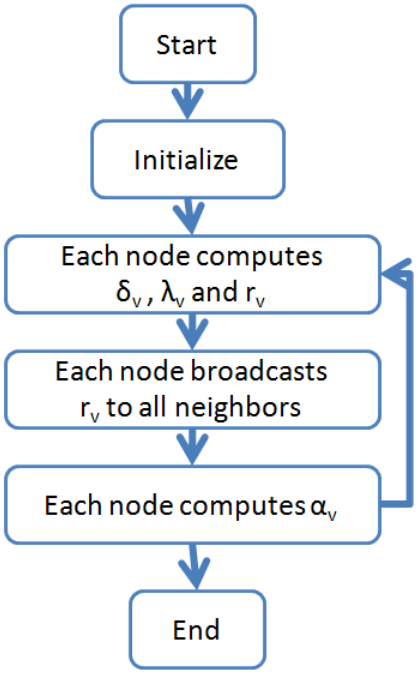


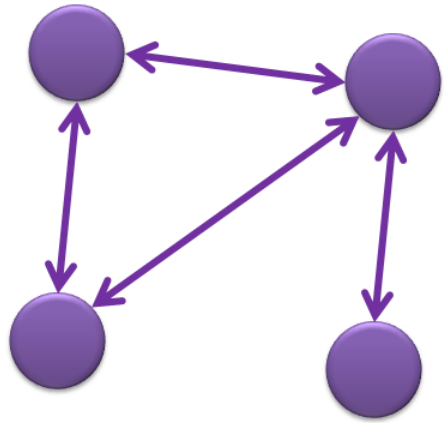


$$\alpha_v(t+1) = \alpha_v(t) + \frac{\eta}{2} \sum_{u \in \mathcal{B}_v} [\mathbf{r}_v(t+1) - \mathbf{r}_u(t+1)], \tag{24}$$

- **Return to Step 1, until convergence.**

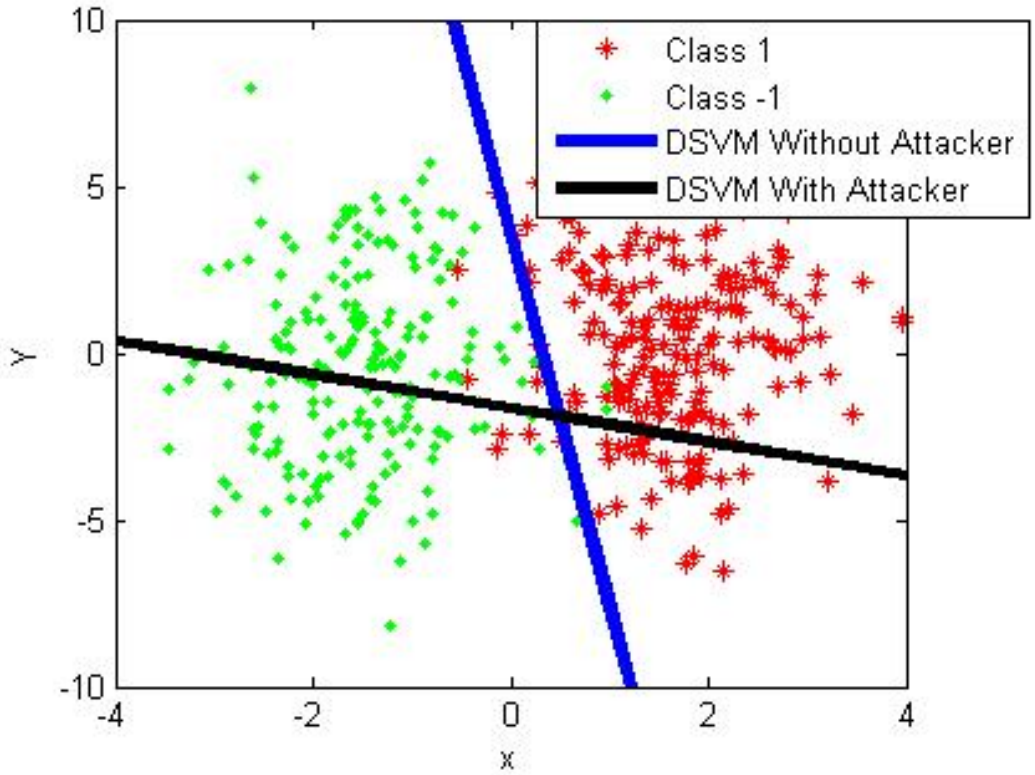
Secure and Resilient Algorithms





- It is **a fully decentralized network operation**, and it does not require exchanging training data or the value of decision functions.
- It provides the **reduced communication overhead** and **privacy preservation** at the same time.

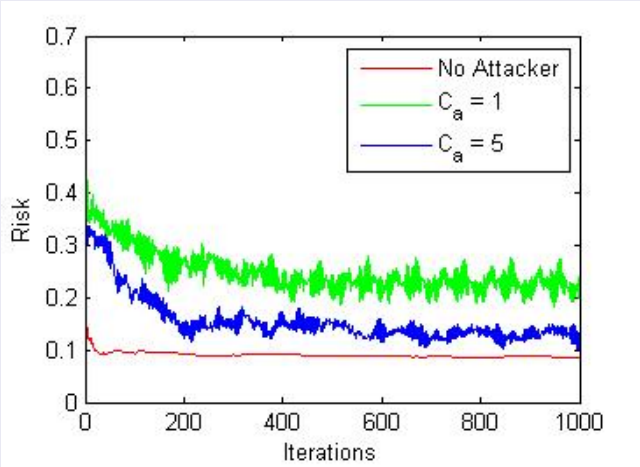
Numerical Experiment: Testing Experiment

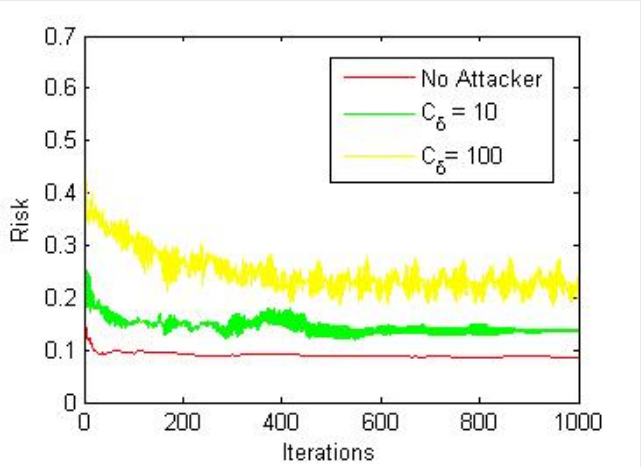


- A network with 4 nodes, each node contains 50 labeled 2-dimension training samples and 50 testing samples from global training set which is shown in this figure.
- The attacker uses the atomic action set with $C_\delta = 100$ and $C_a = 1$.

Effect of Parameters C_a and C_δ

Evolution of the risk of ADMoM-DSVM. Assume that attacker can attack any nodes. Network: 10 nodes, each node has 3 neighbors.

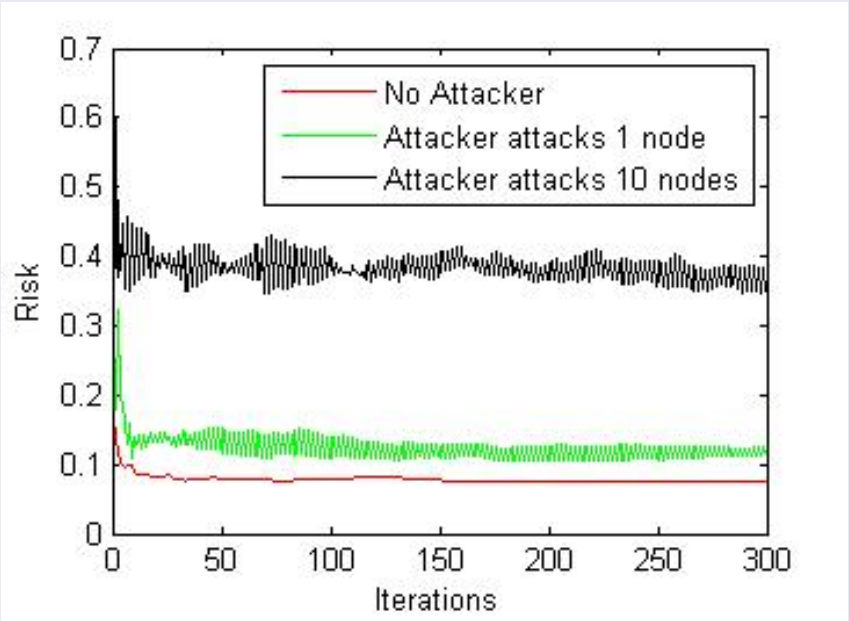




- The left figure shows that the **attacker has less influence** on learner's performance if we **increase the cost** of the attacker (C_a).
- The right figure shows that the **attacker has more influence** on learner if we **increase the size of attacker's action sets** (C_δ).

Effect of the Number of Nodes the Attacker Can Take Over

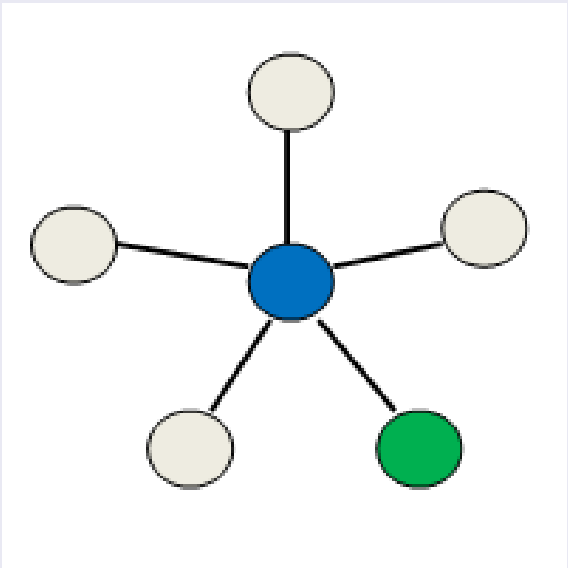
Evolution of the risk under different numbers of nodes the attacker can take over.



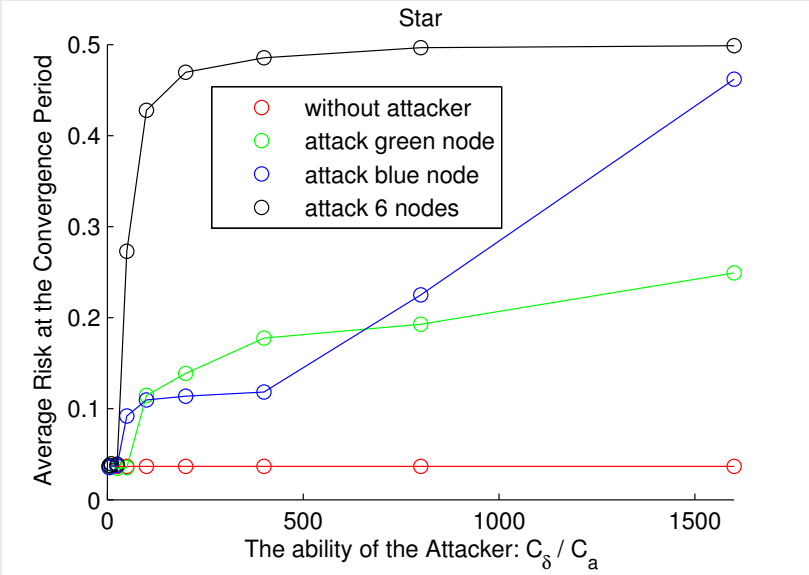
- We can conclude that if the **attacker** has the ability of **taking over more nodes**, he will create a **higher impact** on the learner.

Effect of Network Topology

Average risk in *star* network.



(c) Star



(d) Result:Star

- Fig. (c) shows the network topology.
- Fig. (d) shows the average risk.
- Nodes with **more neighbors (blue)** turn out to have **more influence** on performance as we increase the ability of the attacker (C_δ/C_a).

Conclusions

Contributions

- Capture the attacker's objective and constrained capabilities in a game-theoretic framework.
- Develop a nonzero-sum game to model the strategic interactions between an attacker and a learner with a distributed set of nodes.
- Show the strategic equivalence between the original nonzero-sum game and a zero-sum game.
- Demonstrate that network topology plays an important role in resilience to adversary behaviors.

Future works

- Design network topologies to achieve a desirable level of resiliency.
- Extend to distributed nonlinear SVMs, and other machine learning algorithms.