HW1: Shabani - QM2 (Due Thu Feb 21st)

1. Assume the Hamiltonian of a quantum two-level system in form of $\widehat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}$. The Hamiltonian is Hermitian with real $H_{11}\ and\ H_{22}\ such\ that\ \ H_{11} > H_{22}$.

a. Show that you can write $\widehat{H}$ in the form of $\widehat{H} = \widehat{H_0} + \widehat{H_1}$ where $\widehat{H_0}$ is matrix with a global energy shift and $\widehat{H_1}$ is in form of $\widehat{H_1} = \begin{pmatrix} \epsilon & \Delta - i\widetilde{\Delta} \\ \Delta + i\widetilde{\Delta} & -\epsilon \end{pmatrix}$

b. Find eigenvalues of $\widehat{H_0}$ and $\widehat{H_1}$.

c. Express $\widehat{H_1}$ in terms of Pauli matrices.

d. What is the relationship between eigenstates of $\widehat{H_0}$ ($|\varphi+\rangle, |\varphi-\rangle$) and $\widehat{H_1}$ ($|\psi+\rangle, |\psi-\rangle$). Hint: first solve this for spin ½ system assuming $\widehat{H_1} = -\gamma\hbar\boldsymbol{B}.\mathbf{S}$ where γ is gyromagnetic ratio and $B = (B_x, B_y, B_z)^T$ is the magnetic field vector. Then try to see what is the relationship of $|\varphi+\rangle, |\varphi-\rangle$ to $|\psi+\rangle, |\psi-\rangle$ on the Bloch sphere.

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2. A quantum system is said to possess a 'symmetry' if the Hamiltonian operator, H, is invariant under the associated transformation. In other words, if $H' = H$, where $H' = U^{\dagger}HU$.

a. Show that $H' = H$ is equivalent to $[H, U] = 0$.

b. If a system possesses 'translational symmetry' what operator is a constant of motion?

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3. Consider a particle described by the Hamiltonian $\widehat{H} = \frac{\hat{P}^2}{2m} + \hat{V}(x)$ where $V(x) = \frac{1}{2}m\omega^2x^2 + mgx$. Show that $X' = U^{\dagger}XU = X + d$. (Remember $U|x\rangle = |x - d\rangle$).

a. Solve for d and E_0 such that $H' = U^{\dagger}HU$ satisfies $\widehat{H'} = E_0 + \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega^2x^2$

b. Let $|\varphi_n\rangle$ be eigenstate of H and $|\varphi_n'\rangle$ be eigenstate of H'. What is the relationship between φ_n and φ_n'? What is the relationship between E_n and E_n'?