Quantum Mechanics 2
Spring 2019

Homework 1 Problems

1. Assume the Hamiltonian of a quantum two-level system in form of

$$\hat{H} = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}.$$

The Hamiltonian is Hermitian with $H_{11} > H_{22}$.

(a) Show that you can write $\hat{H}$ in the form of $\hat{H} = \hat{H}_0 + \hat{H}_1$ where $\hat{H}_0$ is a matrix with a global energy shift and $\hat{H}_1$ is of the form:

$$\hat{H} = \begin{pmatrix} \epsilon & \Delta - i\tilde{\Delta} \\ \Delta + i\tilde{\Delta} & -\epsilon \end{pmatrix}.$$

Solution: Since $H_{11} > H_{22}$, we can rewrite them in terms of two other energies E and ϵ as $H_{11} = E + \epsilon$ and $H_{22} = E - \epsilon$. H_{21} is a generic complex number and can always be decomposed into real and imaginary parts $H_{21} = \Delta + i\tilde{\Delta}$. Since $\hat{H}$ is Hermitian, $H_{12} = H_{21}* = \Delta - i\tilde{\Delta}$.

(b) Find the eigenvalues of $\hat{H}_0$ and $\hat{H}_1$.

Solution: The eigenvalues of $\hat{H}_0$ are just E with degeneracy two. To find the eigenvalues of $\hat{H}_1$, we solve the eigenvalue problem:

$$\det(\hat{H}_1 - \lambda\mathbf{1}) = (\lambda - \epsilon)(\lambda + \epsilon) - \Delta^2 - \tilde{\Delta}^2 = 0$$

The solutions to this equation are the eigenvalues $\lambda = \pm\sqrt{\Delta^2 + \tilde{\Delta}^2 + \epsilon^2}$.

(c) Express $\hat{H}_1$ in terms of Pauli matrices.

Solution: $\hat{H}_1 = \Delta\sigma_x + \tilde{\Delta}\sigma_y + \epsilon\sigma_z$.

(d) What is the relationship between the eigenstates of $\hat{H}_0(|\phi, \pm\rangle)$ and the eigenstates of $\hat{H}_1(|\psi, \pm\rangle)$. Hint: First solve this for the spin $\frac{1}{2}$ system assuming $\hat{H}_1 = -\gamma\mathbf{B} \cdot \mathbf{S}$, where γ is the gyromagnetic ratio, $\mathbf{B}$ is the magnetic field vector, and $\mathbf{S} = \frac{\hbar}{2}(\sigma_x, \sigma_y, \sigma_z)$. Then try to see what is the relationship between $|\phi, \pm\rangle$ and $|\psi, \pm\rangle$ on the Bloch Sphere.

Solution: $\hat{H}_0$ is a global energy shift so technically, all vectors are eigenvectors of $\hat{H}_0$ with the same eigenvalue E. We take in the problem the basis to be $|\phi, +\rangle = (10)^T$ and $|\phi, -\rangle = (01)^T$. Writing $\hat{H}_1$ in as a magnetic potential energy of a magnetic dipole amounts to a relabeling of $\Delta, \tilde{\Delta}$ and ϵ but it will give physical insight and help us right the eigenbasis $|\psi, \pm\rangle$ in terms of the Bloch sphere angles θ, ϕ. The relabeling is as follows: $\Delta = -\gamma\frac{hbar}{2}B_x, \tilde{\Delta} = -\gamma\frac{hbar}{2}B_y, \epsilon = -\gamma\frac{hbar}{2}B_z$. In these new variables to eigenvalues are $\lambda = \pm\gamma\frac{hbar}{2}B$ where $B = \sqrt{B_x^2 + B_y^2 + B_z^2}$. This suggests writing the magnetic field in spherical coordinates $\mathbf{B} = (B\cos(\phi)\sin(\theta), B\sin(\phi)\sin(\theta), B\cos(\theta))$ in which case the matrix $\hat{H}_1$ becomes:

$$\hat{H}_1 = -\gamma B\frac{\hbar}{2}\begin{pmatrix} \cos(\theta) & e^{-i\phi}\sin(\theta) \\ e^{i\phi}\sin(\theta) & -\cos(\theta) \end{pmatrix}.$$

In otherwords, the magnitude of the eigenvalues pulls out as a factor and the matrix in terms of Bloch angles has eigenvalues ± 1. Solving for the eigenvectors of this matrix, we find:

$$|\psi, +\rangle = \cos(\frac{\theta}{2})|\phi, +\rangle + e^{i\phi}\sin(\frac{\theta}{2})|\phi, -\rangle$$
$$|\psi, -\rangle = e^{-i\phi}\sin(\frac{\theta}{2})|\phi, +\rangle - \cos(\frac{\theta}{2})|\phi, -\rangle$$

2. A quantum system is said to possess a 'symmetry' if the Hamiltonian operator, H, is invariant under the associated transformation. In other words, if $H' = H$, where $H' \equiv U^\dagger H U$.

   (a) Show that $H' = H$ if and only if $[H, U] = 0$

   Solution: Let's start with the right side. $[H, U] = 0 \Rightarrow HU = UH \Rightarrow U^\dagger HU = U^\dagger UH \Rightarrow U^\dagger HU = H \Rightarrow H' = H$.

   (b) If a system possesses 'translational symmetry' what operator is a constant of motion.

   Solution: Short answer: momentum. Longer answer: Translational symmetry means the Hamiltonian commutes with the translation unitary operator. Since the translation operator is generated by the momentum operator, the Hamiltonian must commute with momentum. Since the Hamiltonian is the generator of time translations, this implies momentum commutes with the time translation operator. This means that momentum remains constant in time (aka, momentum is a constant of motion).

3. Consider a particle described by the Hamiltonian $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + mg\hat{x}$. Show that $\hat{x}' = U^\dagger \hat{x} U = \hat{x} + d$. (Remember $U|x\rangle = |x + d\rangle$).

   Solution: Starting with $U|x\rangle = |x + d\rangle$, we can act on both sides with $\hat{x}$ and we get: $\hat{x}U|x\rangle = (x + d)|x + d\rangle$. Next, we act on both sides with $U^\dagger$ to get: $\hat{x}'|x\rangle = (x + d)|x\rangle = (\hat{x} + d)|x\rangle$. Since $\hat{x}'$ and $\hat{x} + d$ act in the same way on all kets $|x\rangle$, they must be identical operators.

   (a) Solve for d and E_0 such that $H' = E_0 + \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2$.

   Solution: First, when we calculate H' for an arbitrary translation d, we get:

$$
\begin{aligned}
H' &= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2(\hat{x} + d)^2 + mg(\hat{x} + d) \\
&= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 + (m\omega^2 d + mg)\hat{x} + (mgd + \frac{1}{2}m\omega^2 d^2)
\end{aligned}
$$

   The linear term must be zero and from this we get that $d = -\frac{g}{\omega^2}$. The last term is E_0. Plugging in our value for d, we get $E_0 = -\frac{mg^2}{2\omega^2}$.

   (b) Let $|\phi_n\rangle$ be the eigenstates fo H and $|\phi'_n\rangle$ be the eigenstates fo H'. What is the relationship between $|\phi_n\rangle$ and $|\phi'_n\rangle$? Between E_n and E'_n?

   Solution: Let us start with the eigenvalue/eigenvector relationship before the transformation:

$$
H|\phi_n\rangle = E_n|\phi_n\rangle
$$

   We can act on both sides with $U^\dagger$:

$$U^\dagger H|\phi_n\rangle = E_n U^\dagger|\phi_n\rangle$$

Now, we insert $\mathbf{1} = UU^\dagger$ just to the right of H and get:

$$(U^\dagger H U)(U^\dagger|\phi_n\rangle) = E_n(U^\dagger|\phi_n\rangle)$$

Notice however that this is just the eigenvalue/eigenvector relationship for $H' = U^\dagger H U$. Thus, from this equation, we read off the relationships: $|\phi'_n\rangle = U^\dagger|\phi_n\rangle$ and $E'_n = E_n$.