Quantum Mechanics 2
Spring 2019

Homework 5 Problems

1. In Yukawa's original theory (1934) which remains a useful approximation in nuclear physics, the "strong" force between the protons and neutrons is mediated by the exchange of π-mesons (otherwise known as pions). The potential energy is:

$$V(r) = -r_0 V_0 \frac{e^{-\frac{r}{r_0}}}{r} \tag{1}$$

Here r is the distance between the nucleons, and the range r_0 is related to the mass of the meson: $r_0 = \frac{\hbar}{m_\pi c}$. The Schrodinger equation for the proton/neutron system is:

$$-\frac{\hbar^2}{2\mu}\nabla^2\psi(r) + V(r)\psi(r) = E\psi(r) \tag{2}$$

Here μ is the reduced mass of the nucleons. Show that there exists a solution with negative energy (a bound state), using a variational trial wave function of the form: $\psi_B(r) = Ae^{-B\frac{r}{r_0}}$.

   (a) Determine A by normalizing $\psi_B(r)$. Note this is a 3D problem.

   Solution: Our normalization condition is:

$$\int_0^\infty dr 4\pi r^2 \psi_B(r)^2 = \int_0^\infty dr 4\pi A^2 r^2 e^{-2B\frac{r}{r_0}} \Rightarrow A^2 = \frac{B^3}{\pi r_0^3} \tag{3}$$

   (b) Find the expectation value of the Hamiltonian in the state ψ_B. Express your answer in terms of $\gamma = \frac{2\mu r_0^2}{\hbar^2}V_0$

Solution: Let us do the usual trick of defining a modified wavefunction $u_B(r) = r\psi_B(r)$. In terms of this function, the expectation value of the energy is:

$$\begin{aligned} E_B &= 4\pi \int_0^\infty dr \Big[-\frac{\hbar^2}{2\mu} u_B u_B'' - \frac{r_0}{r} V_0 e^{-\frac{r}{r_0}} u_B^2 \Big] \\ &= 4\pi \int_0^\infty dr \Big[-\frac{\hbar^2 B}{2\mu r_0} \Big(\frac{B}{r_0} - \frac{2}{r} \Big) - \frac{r_0}{r} V_0 e^{-\frac{r}{r_0}} \Big] u_B^2 \end{aligned} \tag{4}$$

In the second line, we just used the fact that $u_B'' = \frac{B}{r_0}(\frac{B}{r_0} - \frac{2}{r})u_B$. The first term in the parenthesis coming from the kinetic term when integrated gives us $-\frac{\hbar^2}{2\mu}\frac{B^2}{r_0^2}$ since our wavefunction is already normalized (i.e. $4\pi \int_0^\infty u_B^2 dr = 1$). The second term in parenthesis gives:

$$\begin{aligned} 4\pi \int_0^\infty dr \frac{\hbar^2 B}{2\mu r_0} \frac{2}{r} u_B^2 &= 4\pi A^2 \int_0^\infty dr \frac{\hbar^2 B}{\mu r_0} r e^{-2B\frac{r}{r_0}} \\ &= \frac{4\hbar^2 B^4}{\mu r_0^2} \int_0^\infty du u e^{-2Bu} \\ &= \frac{\hbar^2 B^2}{\mu r_0^2} \end{aligned} \tag{5}$$

In the second line, we plugged in the value for A^2 and used the substitution $u = \frac{r}{r_0}$. The last line involved 2 integration by parts. The potential term is computed as follows:

$$\begin{aligned} 4\pi \int_0^\infty dr \Big[-\frac{r_0}{r} V_0 e^{-\frac{r}{r_0}} u_B^2 \Big] &= 4\pi A^2 \int_0^\infty dr \Big[-\frac{r_0}{r} V_0 e^{-\frac{r}{r_0}} r^2 e^{-2B\frac{r}{r_0}} \Big] \\ &= -4\pi A^2 r_0^3 V_0 \int_0^\infty dx x e^{-(1+2B)x} \\ &= -4\pi A^2 r_0^3 \frac{V_0}{1+2B} \int_0^\infty dx e^{-(1+2B)x} \\ &= -4\pi A^2 r_0^3 \frac{V_0}{(1+2B)^2} \\ &= -\frac{4V_0 B^3}{(1+2B)^2} \end{aligned} \tag{6}$$

In the second line, we made the substitution $x = \frac{r}{r_0}$. In the subsequent 2 lines following this one, we integrated by parts. In the final line, we plugged in the value of A we obtained previously. Putting the results together we get:

$$\begin{aligned} E_B &= \frac{\hbar^2}{2\mu}\frac{B^2}{r_0^2} - \frac{4V_0B^3}{(1+2B)^2} \\ &= V_0\Big(\frac{B^2}{\gamma} - \frac{4B^3}{(1+2B)^2}\Big) \end{aligned} \tag{7}$$

(c) Optimize your wave function by setting $\frac{dE_B}{dB} = 0$. Find E_{min} by eliminating γ in favor of B and express E_{min} as a function of B.

Solution: Setting $\frac{dE_B}{dB} = 0$ gives the following equation:

$$V_0\Big(\frac{2B}{\gamma} - \frac{12B^2}{(1+2B)^2} + \frac{16B^3}{(1+2B)^3}\Big) = 0 \tag{8}$$

We can then solve for γ in terms of B:

$$\gamma = \frac{(1+2B)^3}{4B^2+6B} \tag{9}$$

Plugging this back into the energy, we get:

$$E_B = -V_0\frac{2B^3(1-2B)}{(1+2B)^3} \tag{10}$$

Let us put some numbers on these figures. First we compute γ:

$$\begin{aligned}\gamma = \frac{2\mu}{\hbar^2} r_0^2 V_0 &= \frac{M_N}{\hbar^2}\frac{\hbar}{m_\pi c}(r_0 V_0) \\ &= \frac{M_N}{\hbar^2}\frac{\hbar}{m_\pi c}(\frac{\hbar g_\pi^2}{4\pi c}) \\ &= \frac{M_N}{m_\pi}(\frac{g_\pi^2}{4\pi}) \\ &= \frac{938MeV}{140MeV}(13.55) \\ &\approx 91.1 \end{aligned} \tag{11}$$

We can use this value to determine that $B \approx 3.07$. Plugging this back into the energy, we get:

$$E_B = -1.2GeV \tag{12}$$

2. Bonus