1.2 Hamilton–Jacobi Equation

In general, we can regard the action a function of the final position q_i and time t, keeping the intial data fixed. Then we can show

$$\frac{\partial S}{\partial q_i} = p_i, \qquad \frac{\partial S}{\partial t} = -H. \tag{7}$$

(Here, we already see the connection between the momentum and space-derivative, and the energy and the time-derivative, hinting at what we do in quantum mechanics.) Then one can write the Hamilton–Jacobi equation

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0 \tag{8}$$

using the Hamiltonian $H(p, q)$.

Here is how we see Eq. (7). First of all, when we change the end point of the motion $q_i(t_f)$ to $q_i(t_f)+\delta q_i$, the entire trajectory is changed to $q_i(t)+\delta q_i(t)$ with the boundary conditions $\delta q_i(t_i) = 0$, $\delta q_i(t_f) = \delta q_i$. Remember we evaluate the action along the trajectory that satisifes the equation of motion. The action changes by

$$\begin{aligned} \delta S &= \int_{t_i}^{t+f} \left(\frac{\partial L}{\partial q_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt \\ &= \int_{t_i}^{t+f} \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\delta q_i + \frac{\partial L}{\partial \dot{q}_i}\delta \dot{q}_i\right) dt \\ &= \left[\frac{\partial L}{\partial \dot{q}_i}\delta q_i\right]_{t_i}^{t_f} \\ &= \frac{\partial L}{\partial \dot{q}_i}(t_f)\delta q_i. \end{aligned} \tag{9}$$

In the second one we used the equation of motion. Therefore, we find

$$\frac{\partial S}{\partial q_i} = p_i(t_f). \tag{10}$$

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The variation with respect to t_f needs to be done carefully. When we fix the end point of the motion q_i but change the arrival time t_f to $t_f + \delta t$, we need to change $q_i(t_f)$ to $q_i(t_f) - \dot{q}_i \delta t$ so that it arrives at the same q_i at time $t_f + \delta t$. Therefore, there are two contributions to δS. One is just because of the change in the end point of the time integral $L(t_f)\delta t$, and the other due to the change in $q_i(t_f)$, and hence

$$\delta S = L(t_f)\delta t + \frac{\partial L}{\partial \dot{q}_i}(t_f)(-\dot{q}_i \delta t) = -H\delta t. \tag{11}$$

This proves Eq. (7).