Two-Dimensional Fourier Transform and Linear Filtering

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Outline

• General concept of signals and transforms
  – Representation using basis functions
• Continuous Space Fourier Transform (CSFT)
  – 1D -> 2D
  – Concept of spatial frequency
• Discrete Space Fourier Transform (DSFT) and DFT
  – 1D -> 2D
• Continuous and discrete space convolution
• Convolution theorem
• Applications in image processing
  – Noise removal
  – Edge detection
  – Image sharpening
Signal in 2D Space

• General 2D continuous space signal: \( f(x,y) \)
  – Can have infinite support: \( x,y = (-\infty, \ldots, \infty) \)
  – \( f(x,y) \) can generally take on complex values

• General 2D discrete space signal: \( f(m,n) \)
  – Can have infinite support: \( m,n = -\infty, \ldots, 0, 1, \ldots, \infty \)
  – \( f(m,n) \) can generally take on complex values

• Each color component of an image is a 2D real signal with finite support
  – MxN image: \( m=0,1,\ldots,M-1, n=0,1,\ldots,N-1 \)
  – We will use first index for row, second index for column
  – We will consider a single color component only
  – Same operations can be applied to each component

• Separability
  – \( f(m,n) \) is separable if \( f(m,n) = f_v(m) f_h(n) \)
  – Rank 1 matrix = product of 1D column vector and 1D row vector
Example of Separable (rank 1) Matrices

\[ H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T \]
Transform Representation of Signals

- Transforms are decompositions of a function $f(x)$ into some basis functions $\mathcal{O}(x, u)$. $u$ is typically the freq. index.

**Figure 4.1** The function at the bottom is the sum of the four functions above it. Fourier’s idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.
Illustration of Decomposition in Vector Space

\[ f = \alpha_1 \Phi_1 + \alpha_2 \Phi_2 + \alpha_3 \Phi_3 \]
Representation of 1D Signal Using An Orthonormal Basis

• Orthonormal basis function

\[ \int_{-\infty}^{\infty} \phi(x,u_1) \phi^*(x,u_2) dx = \begin{cases} 1, & u_1 = u_2 \\ 0, & u_1 \neq u_2 \end{cases} \]

• Forward transform

\[ F(u) = \langle f(x), \phi(x,u) \rangle = \int_{-\infty}^{\infty} f(x) \phi^*(x,u) dx \]

• Inverse transform

\[ f(x) = \int_{-\infty}^{\infty} F(u) \phi(x,u) du \]

Projection of \( f(x) \) onto \( \phi(x,u) \) = inner product of \( f(x) \) and \( \phi(x,u) \)

Representing \( f(x) \) as sum of \( \phi(x,u) \) for all \( u \), with weight \( F(u) \)
1D Continuous Time Fourier Transform

• Basis functions (complex sinusoidal or exponential)

\[ \phi(x, u) = e^{j2\pi ux}, \quad u \in (-\infty, +\infty). \]

u = frequency = # cycles per unit of x

\[ \omega = 2\pi u \text{ (radian freq.)} \]

• Forward Transform

\[ F(u) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{-j2\pi ux} \, dx \]

• Inverse Transform

\[ f(x) = F^{-1}\{F(u)\} = \int_{-\infty}^{\infty} F(u)e^{j2\pi ux} \, du \]
Important Transform Pairs

\[
f(x) = 1 \iff F(u) = \delta(u)
\]
\[
f(x) = e^{j2\pi f_0 x} \iff F(u) = \delta(u - f_0)
\]
\[
f(x) = \cos(2\pi f_0 x) \iff F(u) = \frac{1}{2} \left( \delta(u - f_0) + \delta(u + f_0) \right)
\]
\[
f(x) = \sin(2\pi f_0 x) \iff F(u) = \frac{1}{2j} \left( \delta(u - f_0) - \delta(u + f_0) \right)
\]
\[
f(x) = \begin{cases} 
1, & |x| < x_0 \\
0, & \text{otherwise}
\end{cases} \iff F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \text{sinc}(2x_0 u)
\]

where, \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}
FT of the Rectangle Function

\[ F(u) = \frac{\sin(2\pi x_0 u)}{\pi u} = 2x_0 \text{sinc}(2x_0 u) \quad \text{where,} \quad \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \]

Note first zero occurs at \( u_0 = 1/(2x_0) = 1/\text{pulse-width} \), other zeros are multiples of this.
IFT of Ideal Low Pass Signal

• What is $f(x)$?
Representation of FT

- Generally, both $f(x)$ and $F(u)$ are complex
- Two representations
  - Real and Imaginary
  - Magnitude and Phase

$$F(u) = R(u) + jI(u)$$

$F(u) = A(u)e^{j\phi(u)}$, where

$$A(u) = \sqrt{R(u)^2 + I(u)^2}, \quad \phi(u) = \tan^{-1} \frac{I(u)}{R(u)}$$

- Relationship

$$R(u) = A(u)\cos\phi(u), \quad I(u) = A(u)\sin\phi(u)$$

- Power spectrum

$$P(u) = A(u)^2 = F(u) \times F(u)^* = |F(u)|^2$$

- $A(u)$

- $\phi(u)$
What if $f(x)$ is real?

- Real world signals $f(x)$ are usually real
- $F(u)$ is still complex, but has special properties

\[
F^*(u) = F(-u) \\
R(u) = R(-u), A(u) = A(-u), P(u) = P(-u): \text{even function} \\
I(u) = -I(-u), \phi(u) = -\phi(-u): \text{odd function}
\]
Property of Fourier Transform

• Duality
  \[ f(t) \Leftrightarrow F(u) \]
  \[ F(t) \Leftrightarrow f(-u) \]

• Linearity
  \[ F\{a_1 f_1(x) + a_2 f_2(x)\} = a_1 F\{f_1(x)\} + a_2 F\{f_2(x)\} \]

• Scaling
  \[ F\{af(x)\} = aF\{f(x)\} \]

• Translation
  \[ f(x - x_0) \Leftrightarrow F(u)e^{-j2\pi x_0 u}, \quad f(x)e^{j2\pi u_0 x} \Leftrightarrow F(u - u_0) \]

• Convolution
  \[ f(x) \otimes g(x) = \int f(x - \alpha)g(\alpha)d\alpha \]
  \[ f(x) \otimes g(x) \Leftrightarrow F(u)G(u) \]

We will review convolution later!
Two Dimension Continuous Space Fourier Transform (CSFT)

• Basis functions (separable functions)

\[ \phi(x, y; u, v) = e^{j(2\pi ux + 2\pi vy)} = e^{j2\pi ux}e^{j2\pi vy}, \quad u, v \in (-\infty, +\infty). \]

• Forward – Transform

\[ F(u, v) = F\{ f(x, y) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)e^{-j2\pi(ux+vy)} \, dx \, dy \]

• Inverse – Transform

\[ f(x, y) = F^{-1}\{ F(u, v) \} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, v)e^{j2\pi(ux+vy)} \, du \, dv \]

– Representing a 2D signal as sum of 2D complex exponential signals
Illustration of Spatial Frequency

\[ f(x, y) = \sin(10\pi x) \]
\[ f_x = 5, f_y = 0, f_m = 5, \phi = 0 \]
\[ f_x : \text{unit} = 1 \text{ cycle/image-width} \]
\[ f_y : \text{unit} = 1 \text{ cycle/image-height} \]

\[ f(x, y) = \sin(10\pi x - 20\pi y) \]
\[ f_x = 5, f_y = -10, f_m = \sqrt{125}, \phi = \tan(2) \]

fx and fy are used to represent the spatial freq. instead of u,v.
Example 1

\[ f(x, y) = \sin 4\pi x + \cos 6\pi y \]

\[ F\{\sin 4\pi x\} = \int \int \sin 4\pi xe^{-j2\pi(ux+vy)} \, dx \, dy \]

\[ = \int \sin 4\pi xe^{-j2\pi ux} \, dx \int e^{-j2\pi vy} \, dy \]

\[ = \int \sin 4\pi xe^{-j2\pi ux} \, dx \delta(v) \]

\[ = \frac{1}{2j}(\delta(u - 2) - \delta(u + 2))\delta(v) \]

\[ = \frac{1}{2j}(\delta(u - 2, v) - \delta(u + 2, v)) \]

where \( \delta(x, y) = \delta(x)\delta(y) = \begin{cases} \infty, & x = y = 0 \\ 0, & \text{otherwise} \end{cases} \)

Likewise, \( F\{\cos 6\pi y\} = \frac{1}{2}(\delta(u, v - 3) + \delta(u, v + 3)) \)
Example 2

\[f(x, y) = \sin(2\pi x + 3\pi y) = \frac{1}{2j} \left(e^{j(2\pi x + 3\pi y)} - e^{-j(2\pi x + 3\pi y)}\right)\]

\[F\{e^{j(2\pi x + 3\pi y)}\} = \int \int e^{j(2\pi x + 3\pi y)} e^{-j2\pi(xu + yv)} dx dy\]
\[= \int e^{j2\pi x} e^{-j2\pi xu} dx \int e^{j3\pi y} e^{-j2\pi yv} dy\]
\[= \delta(u - 1)\delta(v - \frac{3}{2}) = \delta(u - 1, v - \frac{3}{2})\]

Likewise, \[F\{e^{-j(2\pi x + 3\pi y)}\} = \delta(u + 1, v + \frac{3}{2})\]

Therefore,
\[F\{\sin(2\pi x + 3\pi y)\} = \frac{1}{2j} \left(\delta(u - 1, v - \frac{3}{2}) - \delta(u + 1, v + \frac{3}{2})\right)\]
$f(x, y) = \sin(10 \pi x)$
$f_x = 5, f_y = 0, f_m = 5, \phi = 0$
$f_x$ : unit= 1 cycle/image-width
$f_y$ : unit=1 cycle/image-height

What is its 2D FT?
Properties of 2D FT (1)

- **Linearity**
  
  \[ F\{a_1 f_1(x, y) + a_2 f_2(x, y)\} = a_1 F\{f_1(x, y)\} + a_2 F\{f_2(x, y)\} \]

- **Translation**
  
  \[
  f(x - x_0, y - y_0) \iff F(u, v)e^{-j2\pi(x_0u + y_0v)} ,
  
  f(x, y)e^{j2\pi(u_0x + v_0y)} \iff F(u - u_0, v - v_0) \]

- **Conjugation**

  \[ f^*(x, y) \iff F^*(-u, -v) \]
Properties of 2D FT (2)

• Symmetry

\[ f(x, y) \text{ is real } \iff |F(u, v)| = |F(-u, -v)| \]

• Convolution
  – Definition of convolution

\[ f(x, y) \otimes g(x, y) = \int \int f(x - \alpha, y - \beta) g(\alpha, \beta) d\alpha d\beta \]

  – Convolution theory

\[ f(x, y) \otimes g(x, y) \iff F(u, v)G(u, v) \]

We will describe 2D convolution later!
Separability of 2D FT and Separable Signal

• Separability of 2D FT

\[ F_2 \{f(x, y)\} = F_y \{F_x \{f(x, y)\}\} = F_x \{F_y \{f(x, y)\}\} \]

– where \( F_x \) and \( F_y \) are 1D FT along \( x \) and \( y \).
– one can do 1D FT for each row of original image, then 1D FT along each column of resulting image

• Separable Signal

– \( f(x, y) = f_x(x)f_y(y) \)
– \( F(u, v) = F_x(u)F_y(v) \),
  • where \( F_x(u) = F_x\{f_x(x)\} \), \( F_y(u) = F_y\{f_y(y)\} \)
– For separable signal, one can simply compute two 1D transforms and take their product!
Example 1

\[ f(x, y) = \sin(3\pi x) \cos(5\pi y) \]

\[ f_x(x) = \sin(3\pi x) \quad \Leftrightarrow \quad F_x(u) = \frac{1}{2j} \left( \delta(u - 3/2) - \delta(u + 3/2) \right) \]

\[ f_y(y) = \cos(5\pi y) \quad \Leftrightarrow \quad F_y(v) = \frac{1}{2} \left( \delta(v - 5/2) + \delta(v + 5/2) \right) \]

\[ F(u, v) = F_x(u) F_y(v) \]

\[ = \frac{1}{4j} \left( \delta(u - \frac{3}{2}, v - \frac{5}{2}) - \delta(u + \frac{3}{2}, v - \frac{5}{2}) + \delta(u - \frac{3}{2}, v + \frac{5}{2}) - \delta(u + \frac{3}{2}, v + \frac{5}{2}) \right) \]
Example 2

\[ f(x, y) = \begin{cases} 1, & |x| \leq x_0, \ |y| \leq y_0 \\ 0, & \text{otherwise} \end{cases} \implies \]

\[ F(u, v) = 4x_0y_0 \text{sinc}(2x_0u) \text{sinc}(2y_0v) \]

Is this signal separable?

w/ contrast enhancement through log mapping
Important Transform Pairs

- All following signals are separable and the relations can be proved by applying the separability of the CSFT:

\[
f(x, y) = 1 \iff F(u, v) = \delta(u, v), \quad \text{where} \delta(u, v) = \delta(u)\delta(v)
\]

\[
f(x) = e^{i2\pi(f_1x + f_2y)} \iff F(u) = \delta(u - f_1, v - f_2)
\]

\[
f(x, y) = \cos(2\pi(f_1x + f_2y)) \iff F(u) = \frac{1}{2}(\delta(u - f_1, v - f_2) + \delta(u + f_1, v + f_2))
\]

\[
f(x, y) = \sin(2\pi(f_1x + f_2y)) \iff F(u) = \frac{1}{2j}(\delta(u - f_1, v - f_2) - \delta(u + f_1, v + f_2))
\]

\[
f(x, y) = \begin{cases} 1, & |x| < x_0, |y| < y_0 \\ 0, & \text{otherwise} \end{cases} \iff
\]

\[
F(u, v) = \frac{\sin(2\pi x_0 u)}{\pi u} \frac{\sin(2\pi y_0 v)}{\pi v} = 4x_0y_0 \frac{\text{sinc}(2x_0 u)}{u} \frac{\text{sinc}(2y_0 v)}{v}
\]

where \( \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \)

\[
\exp \left\{ -\frac{x^2 + y^2}{2\sigma^2} \right\} \iff \exp \left\{ -\frac{u^2 + v^2}{2\beta^2} \right\}, \quad \beta = \frac{1}{2\pi\sigma}
\]

- Constant \( \iff \) impulse at (0,0) freq.

- Complex exponential \( \iff \) impulse at a particular 2D freq.

- 2D box \( \iff \) 2D sinc function

- Gaussian \( \iff \) Gaussian
Rotation

- Let \( x = r \cos \theta, \ y = r \sin \theta, \ u = \rho \cos \omega, \ v = \rho \sin \omega. \)

- 2D FT in polar coordinate \((r, \theta)\) and \((\rho, \phi)\)

\[
F(\rho, \phi) = \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \theta) e^{-j2\pi (r \cos \theta \rho \cos \phi + r \sin \theta \rho \sin \phi)} r dr d\theta
\]

\[
= \int \int f(r, \theta) e^{-j2\pi \rho \cos(\theta - \phi)} r dr d\theta
\]

- Property

\[
f(r, \theta + \theta_0) \iff F(\rho, \phi + \theta_0)
\]

- Proof: Homework!

- Note: In this notation: \(x\) is horizontal (column) index, \(y\) is vertical (row)
Example of Rotation

FIGURE 4.25
(a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).

From Gonzalez
1D Fourier Transform For Discrete Time Sequence (DTFT) (Review)

• f(n) is a 1D discrete time sequence

• Forward Transform

\[
F(u) = \sum_{n=-\infty}^{\infty} f(n) e^{-j2\pi un}
\]

• Inverse Transform

\[
f(n) = \int_{-1/2}^{1/2} F(u) e^{j2\pi un} du
\]

• Representing f(n) as a weighted sum of many complex sinusoidal signals with frequency u, F(u) is the weight

• F(u) indicate the “amount” of sinusoidal component with freq. u in signal f(n)

• |u| = digital frequency = the number of cycles per integer sample. Period = 1/|u| (must be equal or greater than 2 samples-> |u|<=1/2)
Properties unique for DTFT

• Periodicity
  – $F(u) = F(u+1)$
  – The FT of a discrete time sequence is only considered for $u \in (-\frac{1}{2}, \frac{1}{2})$, and $u = \pm \frac{1}{2}$ is the highest discrete frequency

• Symmetry for real sequences

\[
f(n) = f^*(n) \iff F(u) = F^*(-u)\]

\[
\Rightarrow |F(u)| = |F(-u)|
\]

\[
\Rightarrow |F(u)| \text{ is symmetric}
\]
Example

\[ f(n) = \begin{cases} 1, & n = 0, 1, \ldots, N - 1; \\ 0, & \text{others} \end{cases} \]

\[ F(u) = \sum_{n=0}^{N-1} e^{-j2\pi nu} = \frac{1 - e^{-j2\pi uN}}{1 - e^{-j2\pi u}} = e^{-j\pi(N-1)u} \frac{\sin 2\pi u(N/2)}{\sin 2\pi u(1/2)} \]

Digital sinc: There are \( N/2 - 1 \) zeros in (0, \( 1/2 \]), 1/N apart
1D Discrete Fourier Transform (DFT)

- **DTFT of N-pt sequence**

\[
F(u) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi un}
\]

- **N-pt DFT of N-pt sequence**

\[
F_N(k) = \sum_{n=0}^{N-1} f(n) e^{-j2\pi \frac{kn}{N}} = F \left( u = \frac{k}{N} \right)
\]

- **DFT** is the sampled version of DTFT with samples at 0, 1/N, ..., (N-1)/N, if f(n) has length <=N

- **FFT**: Fast algorithm for computing DFT
  - Direct computation takes \(N^2\) operations
  - FFT takes \(~N \log(N)\) operations!
Discrete Space Fourier Transform (DSFT) for Two Dimensional Signals

- Let $f(m,n)$ represent a 2D sequence
- Forward Transform
  \[
  F(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m,n)e^{-j2\pi(mu+nv)}
  \]
- Inverse Transform
  \[
  f(m,n) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u,v)e^{j2\pi(mu+nv)} \, du \, dv
  \]
- Representing an image as weighted sum of many 2D complex sinusoidal images
- $u$: number of cycles per vertical sample (vertical freq.)
- $v$: number of cycles per horizontal sample (horizontal freq.)
Spatial Frequency for Digital Images

If the image has 256x256 pixels, \( f_x = 5 \) cycles per width (analog frequency) \( \rightarrow u = \frac{5}{256} \) cycles/sample (digital frequency)

When both horizontal and vertical frequency are non-zero, we see directional patterns. \( f_s \) is the frequency along the direction with the maximum change (orthogonal to the lines)

Note that \( 1/u \) may not correspond to integer.

If \( u = a/b \), digital period = b. Ex. \( u = 3/8 \), Digital Period = 8

\[
f_s = \sqrt{125}, \quad \phi = \arctan(2)
\]
Periodicity

\[ F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m, n)e^{-j2\pi(mu+nv)} \]

• \( F(u,v) \) is periodic in \( u, v \) with period 1, i.e., for all integers \( k, l \):
  - \( F(u+k, v+l) = F(u, v) \)
• To see this consider

\[
e^{-j2\pi(m(u+k)+n(v+l))} = e^{-j2\pi(mu+nv)} e^{-j2\pi(mk+nl)} = e^{-j2\pi(mu+nv)}
\]
Example: Delta Function

- Fourier transform of a delta function

\[ F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(m, n) e^{-j2\pi(mu+nv)} = 1 \]

\[ \delta(m, n) \leftrightarrow 1 \]

- Delta function contains all frequency components with equal weights!

- Inverse Fourier transform of a delta function

\[ f(m, n) = \int_{-0.5}^{0.5} \int_{-0.5}^{0.5} \delta(u, v) e^{j2\pi(mu+nv)} dudv = 1 \]

\[ 1 \leftrightarrow \delta(u, v) \]
Example

\[ f(m, n) = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{bmatrix} \]

\[
F(u, v) = 1e^{-j2\pi(-1*u-1*v)} + 2e^{-j2\pi(-1*u+0*v)} + 1e^{-j2\pi(-1*u+1*v)} \\
-1e^{-j2\pi(1*u-1*v)} - 2e^{-j2\pi(1*u+0*v)} - 1e^{-j2\pi(1*u+1*v)} \\
= j2\sin 2\pi u e^{j2\pi v} + j4\sin 2\pi u + j2\sin 2\pi u e^{-j2\pi v} \\
= j4\sin 2\pi u (\cos 2\pi v + 1)
\]

Note: This signal is low pass in the horizontal direction \( v \), and **band pass** in the vertical direction \( u \).
Graph of $F(u,v)$

```matlab
du = [-0.5:0.01:0.5];
dv = [-0.5:0.01:0.5];
Fu = abs(sin(2 * pi * du));
Fv = cos(2 * pi * dv);
F = 4 * Fu' * (Fv + 1);
mesh(du, dv, F);
colorbar;
Imagesc(F);
colormap(gray); truesize;
```

Using MATLAB `freqz2`:
```
f=[1,2,1;0,0,0;-1,-2,-1];
freqz2(f)
```
DSFT vs. 2D DFT

• 2D DSFT

$$F(u, v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m, n)e^{-j2\pi(mu+nv)}$$

• 2D DFT

$$F(k, l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)e^{-j2\pi(mk/M+nl/N)}$$

• (MxN) point 2D DFT are samples of DSFT for images at u=k/M, v=l/N, as longs M and N are equal to or greater than the height and width of the image, respectively. Can be computed using FFT algorithms. 1D FFT along rows, then 1D FFT along columns.
Display of DFT of Images

- `Imshow(img)`
- `MFimg=abs(fft2(img)), imshow(MFimg,[ ])`
- `SMFimg=fftshift(MFimg)`
- `imshow(SMFimg,[ ])`
- `LSMFimg=log(SMFimg+1); imshow(LSMFimg,[ ])`

- **fftshift** to shift the (0,0) freq. to center
- **Log mapping** to enhance contrast (boost low values!)
DFT of Typical Images

How to interpret the DFT image? Why is it bright at center and has some line structures?

$|F(u,v)|$ (obtained using 2D FFT)
$ff=\text{abs}(\text{fft2}(f));$
$\text{imagesc}(\text{fftshift}(\text{log}(ff+1)));$
Log mapping to enhance contrast
Fftshift to shift the (0,0) freq. to center
Which one below is the DFT of which one above?
DSFT of Separable Signals

- Separable signal:
  - \( h(m,n) = h_x(m) \cdot h_y(n) \)

- 2D DSFT of separable signal = product of 1D DSFT of each 1D component
  - \( H(u,v) = H_x(u) \cdot H_y(v) \)
  - \( H_x(u) \): 1D FT of \( h_x \)
  - \( H_y(v) \): 1D FT of \( h_y \)
DSFT of Special Signals

- Constant $\leftrightarrow$ pulse at $(0,0)$
- Rectangle $\leftrightarrow$ 2D digital sinc
- Sinc $\leftrightarrow$ Rectangle (ideal low pass)
- Complex sinusoid with freq $(u_0, v_0)$ $\leftrightarrow$ pulse at $(u_0, v_0)$
- ...
- Can be shown easily by making use of the fact that the signal is separable
Using Separable Processing to Compute DSFT

• 3x3 averaging filter

\[
H = \frac{1}{9} \begin{bmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{bmatrix}
\]

• Recognizing that the filter is separable

\[
H(u, v) = H_1(u)H_1(v) = \frac{(1 + \cos 2\pi u)(1 + \cos 2\pi v)}{9}
\]

\[
H(v) = \frac{1}{3} (1 + e^{j2\pi v} + e^{-j2\pi v}) = \frac{1 + 2\cos 2\pi v}{3}
\]

\[
H_1(v) = \frac{1}{3} (1 + 2\cos \pi v)
\]
Using Separable Processing to Compute DSFT

- A separable signal (filter)

\[ H = \frac{1}{9} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} \]

- Recognizing that the filter is separable

\[ H = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} = h_x h_y^T \]

Note the constant 1/9 is missing

\[ \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \rightarrow F_y(v) = 1e^{j2\pi v} + 0 + (-1)e^{-j2\pi v} = 2j \sin 2\pi v \]

\[ \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \rightarrow F_x(u) = 1e^{j2\pi u} + 2 + e^{-j2\pi u} = 2 + 2 \cos 2\pi u \]

\[ F(u, v) = F_x(u)F_y(v) = 4j(1 + \cos 2\pi u) \sin 2\pi v \]
What about rotation?

- No theoretical proof, however, roughly it is still true
- Rotation in space <-> Rotation in freq.
- Example:
  - \( f(m,n) = \delta(m) \) (horizontal line) <-> \( F(u,v) = \delta(v) \) (vertical line)
  - What about \( f(m,n) = \delta(m-n) \) (diagonal line)?
    - <-> \( F(u,v) = \delta(u+v) \) (antidiagonal line!)
    - Homework
Linear Convolution over Continuous Space

- 1D convolution

\[
f(x) * h(x) = \int_{-\infty}^{\infty} f(x - \alpha)h(\alpha)d\alpha = \int_{-\infty}^{\infty} f(\alpha)h(x - \alpha)d\alpha
\]

\[
f(x) * \delta(x) = f(x), \quad f(x) * \delta(x - x_0) = f(x - x_0)
\]

- 2D convolution

\[
f(x, y) * h(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - \alpha, y - \beta)h(\alpha, \beta)d\alpha d\beta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\alpha, \beta)h(x - \alpha, y - \beta)d\alpha d\beta
\]
Examples of 1D Convolution

(1) $0 \leq x < 1$

(2) $1 \leq x < 2$
Example of 2D Convolution

1. \( 0 < x \leq 1, 0 < y \leq 1 \)
   \( g(x, y) = x \cdot y \)

2. \( 0 < x \leq 1, 1 < y \leq 2 \)
   \( g(x, y) = x \cdot (2 - y) \)

3. \( 1 < x \leq 2, 0 < y \leq 1 \)
   \( g(x, y) = (2 - x) \cdot y \)

4. \( 1 < x \leq 2, 1 < y \leq 2 \)
   \( g(x, y) = (2 - x) \cdot (2 - y) \)
Convolution of 1D Discrete Signals

- Definition of convolution

\[ f(n) \ast h(n) = \sum_{m=-\infty}^{\infty} f(n-m)h(m) = \sum_{m=-\infty}^{\infty} f(m)h(n-m) \]

- The convolution with \( h(n) \) can be considered as the weighted average in the neighborhood of \( f(n) \), with the filter coefficients being the weights
  - sample \( f(n-m) \) is multiplied by \( h(m) \)
- The filter \( h(n) \) is the impulse response of the system (i.e. the output to an input that is an impulse)
- Signal length before and after filtering
  - Original signal length: \( N \)
  - Filter length: \( K \)
  - Filtered signal length: \( N+K-1 \)
Example of 1D Discrete Convolution

(a) $n < 0$, $g(n) = 0$

(b) $0 \leq n \leq 8$, $g(n) > 0$

(c) $n > 8$, $g(n) = 0$
Convolution of 2D Discrete Signals

Each new pixel $g(m,n)$ is a weighted average of its neighboring pixels in the original image:
- Pixel $f(m-k,n-l)$ is weighted by $h(k,l)$

We may use matrices to represent both signal ($F$) and filter ($H$) and use $F*H$ to denote the convolution
Example of 2D Discrete Convolution

\[
f(m,n) \\
h(m,n) \\
h(-k,-l)
\]

\[
g(-1, -2)=1 \\
g(2,1)=1 \\
g(m,n)
\]
Example: Averaging and Weighted Averaging
Example

Original image
Average filtered image
Weighted Average filtered image
Example: Edge Detection

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
-1 & 0 & 1 \\
-1 & 0 & 1 \\
1 & 0 & -1 \\
1 & 0 & -1 \\
1 & 0 & -1 \\
\end{array}
\]

\[
\begin{array}{ccccc}
100 & 100 & 100 & 100 & 100 \\
100 & 200 & 205 & 203 & 100 \\
100 & 195 & 200 & 200 & 100 \\
100 & 200 & 205 & 195 & 100 \\
100 & 100 & 100 & 100 & 100 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 0 & -1 \\
2 & 0 & -2 \\
1 & 0 & -1 \\
\end{array}
\]

flip
Example of Sobel Edge Detector

Original image

Filtered image by $H_x$

Filtered image by $H_y$

\[
\begin{array}{ccc}
1 & 0 & -1 \\
2 & 0 & -2 \\
1 & 0 & -1 \\
\end{array}
\]
What does h(m,n) mean?

- Any operation that is linear and shift invariant can be described by a convolution with a filter h(m,n)!
- h(m,n) is the impulse response of the system (i.e. output of the system to an impulse input $\delta(m,n)$)
- Better known as **point spread function**, indicating how a single point (i.e. an impulse) in the original image would be spread out in the output image
Point Spread Function

- The point spread function of an imaging system (e.g. a camera or a medical imaging system) describes the resolution of the system:
  - Two object points cannot be separated if they are closer than the support of the point spread function!
Boundary of Filtered Image

- An image of size MxN convolving with a filter of size KxL will yield an image of size (M+K-1,N+L-1)
- If the filter is symmetric with (2k+1)x(2k+1) samples, the convolved image should have an extra boundary of thickness k on each side outside the original image (outer boundary, green). The values along the outer boundary depend on the assumed pixel values outside the original image.
- Filtered values in the inner boundary (orange) of k pixels inside the original image also depend on the assumed pixel values outside the original image.
- Filtered values in the valid region (red) only depend on the available image values.
Boundary Treatment: Zero Padding

- MxN image convolved with KxL filter -> (M+K-1)x(N+L-1) image
- Filtered values at the outer and inner boundary of the output image depend on the assumed value of the pixels outside the original image
- Zero padding: Assuming pixel values are 0 outside the original image
Boundary Treatment: Symmetric Extension

- Assuming pixels values outside the image are the same as their mirroring pixels inside the image
  - Lead to less discontinuity in the filtered image along the outer and inner boundaries

\[
\begin{array}{cccc}
200 & 200 & 205 & 203 \\
200 & 200 & 205 & 203 \\
195 & 195 & 200 & 200 \\
200 & 200 & 205 & 195 \\
200 & 200 & 205 & 195 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array}
\]

Actual image pixels  Extended pixels

\[
\frac{1}{9} \times \begin{array}{cccc}
(200+200+200)/9 & (200+200+205)/9 & \ldots & \ldots \\
(200+200+205)/9 & (200+200+205)/9 & \ldots & \ldots \\
(200+200+205+195+195+200)/9 & (200+200+205+195+195+200)/9 & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
Simplified Boundary Treatment

- Filtered image size = original image size
- Only compute in the valid region.
  - Assign 0 (for high/band pass filters) or keep original value (for low pass) in the inner boundary
- Resulting image is correct only in the “valid” region

```
1 1 1
1 1 1
1 1 1
```

Outer boundary: not considered

```
1/9 * 

1 1 1
1 1 1
1 1 1
```

Inner boundary: not processed
Example: Simplified Boundary Treatment

\[
\begin{array}{c c c c c}
100 & 100 & 100 & 100 & 100 \\
100 & 200 & 205 & 203 & 100 \\
100 & 195 & 200 & 200 & 100 \\
100 & 200 & 205 & 195 & 100 \\
100 & 100 & 100 & 100 & 100 \\
\end{array}
\]
Sample Matlab Program (With Simplified Boundary Treatment)

% readin bmp file
x = imread('lena.bmp');
[xh xw] = size(x);
y = double(x);

% linear convolution, assuming the filter is non-separable (although this example filter is separable)
z = y; %or z=zeros(xh,xw) if not low-pass filter
for m = hhh + 1:xh - hhh,
    %skip first and last hhh rows to avoid boundary problems
    for n = hhw + 1:xw - hhw,
        %skip first and last hhw columns to avoid boundary problems
        tmpy = 0;
        for k = -hhh:hhh,
            for l = -hhw:hhw,
                tmpv = tmpv + y(m - k,n – l)* h(k + hhh + 1, l + hhw + 1);
            end
        end
        z(m, n) = tmpv;
    end
end

% define 2D filter
h = ones(5,5)/25;
[hh hw] = size(h);
hhh = (hh - 1) / 2;
hhw = (hw - 1) / 2;

% for more efficient matlab coding, you can replace the above loop with
z(m,n)=sum(sum(y(m-hhh:m+hhh,n-hhw:n+hhw).*flip(h)))
end
end
Separable Filters

- A filter is separable if \( h(m, n) = h_x(m)h_y(n) \).
- Matrix representation

\[
H = h_x h_y^T
\]

- Where \( h_x \) and \( h_y \) are column vectors
- Example

\[
H_x = \frac{1}{4} \begin{bmatrix}
-1 & -2 & -1 \\
0 & 0 & 0 \\
1 & 2 & 1 \\
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
-1 \\
0 \\
1 \\
\end{bmatrix} \begin{bmatrix}
1 & 2 & 1 \\
\end{bmatrix}, \quad H_y = \frac{1}{4} \begin{bmatrix}
-1 & 0 & 1 \\
-2 & 0 & 2 \\
-1 & 0 & 1 \\
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
1 \\
2 \\
-1 & 0 & 1 \\
\end{bmatrix}
\]
Separable Filtering

• If h(m,n) is separable, the 2D convolution can be accomplished by first applying 1D filtering along each row using h_y(n), and then applying 1D filtering to the intermediate result along each column using the filter h_x(n) (or column filtering followed by row filtering).

• Proof

\[
f(m,n) * h(m,n) = \sum_k \sum_l f(m-k, n-l) h_x(k) h_y(l)
= \sum_k \left( \sum_l f(m-k, n-l) h_y(l) \right) h_x(k)
= \sum_k g_y(m-k, n) h_x(k)
= (f(m,n) * h_y(n)) * h_x(m)
\]
Results Using Sobel Filters

Original image

Filtered image by $H_x$

Filtered image by $H_y$

What do $H_x$ and $H_y$ do?

$H_x = \frac{1}{4} \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$H_y = \frac{1}{4} \begin{bmatrix} -1 & 0 & 1 \\ -2 & 0 & 2 \\ -1 & 0 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$
Computation Cost: Non-Separable vs. Separable Filtering

• Suppose: Image size M*N, filter size K*L. Ignoring outer boundary pixels
• Non-separable filtering
  – Weighted average on each pixel: K*L mul; K*L – 1 add.
  – For all pixels: M*N*K*L mul; M*N*(K*L-1) add.
  – When M=N, K=L: $M^2K^2$ mul + $M^2(K^2-1)$ add.
• Separable filtering:
  – Each pixel in a row: L mul; L-1 add.
  – Each row: N*L mul; N*(L-1) add.
  – M rows: M*N*L mul; M*N*(L-1) add.
  – Each pixel in a column: K mul; K-1 add.
  – Each column: M*K mul; M*(K-1) add.
  – Total: M*N*(K+L) mul; M*N*(K+L-2) add.
  – When M=N, K=L: $2M^2K$ mul; $2M^2(K-1)$ add.
    • Significant savings if K (and L) is large!
MATLAB Function: conv2( )

- Matlab functions
  - C=conv2(H,F,shape)
    - Shape='full' (Default): C includes both outer and inner boundary, using zero padding
    - Shape="same": C includes the inner boundary, using zero padding
    - Shape="valid": C includes the convolved image without the inner boundary, computed without using pixels outside the original image
  - C=conv2(h1,h2,F,shape)
    - Separable filtering with h1 for column filtering and h2 for row filtering
Python Function: `conv2( )`

- Python functions
  - `C=scipy.signal.convolve2d(in1, in2, mode)`
    - `mode` = ‘full’ (Default): C includes both outer and inner boundary, using zero padding
    - Shape=“same”: C includes the inner boundary, using zero padding
    - Shape=“valid”: C includes the convolved image without the inner boundary, computed without using pixels outside the original image
Notes about implementation

- Input image needs to be converted to float or double
  - MATLAB imread and Python cv2.imread return unsigned characters
  - *Never do numerical operations on unsigned character type!*
- Output image value may not be in the range of (0,255) and may not be integers
- To display or save the output image properly
  - Renormalize to (0,255) using a two-pass operation
    - First pass: save directly filtered value in an intermediate floating-point array
    - Second pass: find minimum and maximum values of the intermediate image, renormalize to (0,255) and rounding to integers
      - $F = \text{round}((F1-fmin)*255/(fmax-fmin))$
  - To display the unnormalized image directly in MATLAB, use `imagesc(img)`. Or `imshow(img, [ ])`.
  - In python: `cv2.imshow(‘fig_name’,img), cv2.waitKey()` or `plt.imshow(img)` (check Lecture note 1: ContrastEnhancement page 24)
Convolution Theorem

- Convolution Theorem

\[ f * h \Leftrightarrow F \times H, \quad f \times h \Leftrightarrow F * H \]

- Proof

\[
g(m, n) = f(m, n) * h(m, n) = \sum_k \sum_l f(m - k, n - l)h(k, l)
\]

**FT on both sides**

\[
G(u, v) = \sum_{m,n} \sum_{k,l} f(m - k, n - l)h(k, l)e^{-j2\pi(mu + nv)}
\]

\[
= \sum_{m,n} \sum_{k,l} f(m - k, n - l)e^{-j2\pi((m-k)u+(n-l)v)}h(k, l)e^{-j2\pi(ku + lv)}
\]

\[
= \sum_{m,n} f(m - k, n - l)e^{-j2\pi((m-k)u+(n-l)v)} \sum_{k,l} h(k, l)e^{-j2\pi(ku + lv)}
\]

\[
= \sum_{m',n'} f(m', n')e^{-j2\pi(m'u + n'v)} \sum_{k,l} h(k, l)e^{-j2\pi(ku + lv)}
\]

\[
= F(u, v) \times H(u, v)
\]
Another view of convolution theorem

\[ f(m, n) * h(m, n) \iff F(u, v)H(u, v) \]

- \( F(u, v)H(u, v) = \) Modifying the signal’s each frequency component’ complex magnitude \( F(u, v) \) by \( H(u, v) \)

\[ f(m, n) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u, v)e^{j2\pi(mu+nv)} \, du \, dv \]

\[ g(m, n) = \int_{-1/2}^{1/2} \int_{-1/2}^{1/2} F(u, v)H(u, v)e^{j2\pi(mu+nv)} \, du \, dv \]

- \( H(u, v) \) is also called **Frequency Response** of the 2D LSI system
  - \( \exp{2\pi(um+vn)} \to H(u, v) \exp{2\pi(um+vn)} = |H(u, v)|\exp{2\pi(um+vn)+\varphi(H(u, v))} \)
  - Sinusoid (or complex exponential) input \( \to \) sinusoid (complex exponential) output!
  - \( H(u, v) \) describes how the magnitude and phase of a sinusoid input with frequency \( (u, v) \) are changed!
Explanation of Convolution in the Frequency Domain

\[ g(x) = f(x) * h(x) \]

\[ G(u) = F(u) H(u) \]
Example

- Given a 2D filter, determine its frequency response. Apply to a given image, show original image and filtered image in pixel and freq. domain

\[
h = \frac{1}{25}
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}
\]
$h = \frac{1}{25} \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{bmatrix}$
Matlab Program Used

```matlab
x = imread('lena256.bmp');
figure(1); imshow(x);
f = double(x);
ff = abs(fft2(f));
figure(2); imagesc(fftshift(log(ff+1))); colormap(gray); truesize; axis off;
h = ones(5,5)/9;
hf = abs(freqz2(h));
figure(3); imagesc((log(hf+1))); colormap(gray); truesize; axis off;
y = conv2(f, h);
figure(4); imagesc(y); colormap(gray); truesize; axis off;
yf = abs(fft2(y));
figure(5); imagesc(fftshift(log(yf+1))); colormap(gray); truesize; axis off;
```
\[ H_1 = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ -1 & -1 & -1 \end{bmatrix} \]
Typical Filter Types

- **Low Pass**
- **High Pass**
- **Band Pass**

Non-zero frequency components, where $F(u,v) \neq 0$
Filter Design

• The ideal low-pass, high-pass, band-pass filters not realizable
• Filter design (ECE-GY 6113 DSP1!)
  – Apply optimization technique to find the filter to match the desired frequency response as close as possible under some constraint (e.g. length)
  – FIR vs. IIR filters
  – Linear phase vs. non-linear phase
  – FIR: can realize linear phase
  – IIR filters can realize same transition band/attenuation with much shorter filter order than FIR!
  – Non-linear phase of IIR can be overcome through filtering forward and backward (filtfilt( ))!
• 1D filter design method can be used to design separable 2D filters
  – Not a focus of this class
• Image processing typically use short filters
Calculate Linear Convolution Using DFT

- **1D case**
  - $f(n)$ is length $N_1$, $h(n)$ is length $N_2$
  - $g(n) = f(n) * h(n)$ is length $N = N_1 + N_2 - 1$.
  - To use DFT, need to **extend** $f(n)$ and $h(n)$ to length $N$ by zero padding.
  - $H(k)$ can be precalculated
Computing 2D Convolution Using 2D DFT

Relation between spatial and frequency domain operation:

\[ g(x, y) = h(x, y) \otimes f(x, y) \iff G(u, v) = H(u, v)F(u, v) \]
\[ h(x, y) = \text{IDFT}(H(u, v)), \quad H(u, v) = \text{DFT}(h(x, y)). \]

**FIGURE 4.5** Basic steps for filtering in the frequency domain.
Image Filtering Using DFT

- Typically DFT size=image size. This corresponds to circular convolution, which differs from linear convolution at the inner boundaries. Only correct in the valid region.

- Circular convolution

\[ f(n) \otimes h(n) = \sum_{k=0}^{N-1} f((n-k) \mod N)h(k) \]

\[ f(n) \otimes h(n) \Leftrightarrow F_N(k)H_N(k) \]

- Image filters typically have short length to avoid boundary problems and ringing effect.

- For image filtering with short filters, it is more efficient to do convolution in the spatial domain directly.
Typical Image Processing Tasks

- Noise removal (image smoothing): low pass filter
- Edge detection: high pass filter followed by non-linear processing (e.g. peak detection)
- Image sharpening: high emphasis filter
- ...
- In image processing, we rarely use very long filters
- We compute convolution directly, instead of using 2D FFT
- Filter design: For simplicity we often use separable filters, and design 1D filter based on the desired frequency response in 1D
- We do not focus on filter design in this class
Noise Removal Using Averaging Filters

Window size controls tradeoff between noise removal power and blurring.
Freq. Response Corresponding to Averaging Filters of Different Sizes

```
H = ones(3,3);
Hf = freqz2(H);
figure(1);
mesh(abs(Hf));
title('Freq. Response of 3x3 Averaging Filter');
figure(2);
imshow(abs(Hf), []);
3x3 filter
H = ones(7,7);
Hf = freqz2(H);
figure(1);
mesh(abs(Hf));
title('Freq. Response of 7x7 Averaging Filter');
figure(2);
imshow(abs(Hf), []);
7x7 filter
```
Gaussian Filter in Continuous Space

- FT of Gaussian still a Gaussian Function!
- 1D Gaussian filter
  \[
  \exp \left\{ -\frac{x^2}{2\sigma^2} \right\} \iff \exp \left\{ -\frac{u^2}{2\beta^2} \right\}, \beta = \frac{1}{2\pi\sigma}
  \]
- 2D Gaussian filter
  \[
  \exp \left\{ -\frac{x^2 + y^2}{2\sigma^2} \right\} \iff \exp \left\{ -\frac{u^2 + v^2}{2\beta^2} \right\}, \beta = \frac{1}{2\pi\sigma}
  \]
- Note that STD in freq. $\beta$ inversely related to STD $\sigma$ in space
Illustration of Gaussian Signal (1D)

$\sigma = 1$

$\beta = 0.16$

$\sigma = 2$

$\beta = 0.08$
2D Gaussian Function

Surface plot (surf( ))

Plot as image
Gaussian Filter: from continuous to discrete space

• Analog form: STD $\sigma$ controls the smoothing strength

$$h(x, y) = \alpha \exp\left\{-\frac{x^2 + y^2}{2\sigma^2}\right\},$$

• Take samples, truncate after a few STD, normalize the sum to 1. Usually $\sigma \geq 1$

• Size of mask $K \times K$, $K \geq 5\sigma$, odd
  – Ex: $\sigma = 1$, $K = 7$.
  – Show filter mask,
  – Show frequency response

• Essentially a weighted average filter with decreasing weights away from the center

• A good filter for removing noise. $\sigma$ should be chosen based on noise STD.
Ex: 7x7 Gaussian Filter Generation

function gauss(s)

    x=[-3.0:1.0:3.0];
    gauss=exp(-x.^2/(2*s^2));
    gauss2=gauss'*gauss;
    gauss2=gauss2/(sum(sum(gauss2)));
    H=gauss2;
    disp(H);
    figure(1);
    surf(gauss2);
    figure(2);
    freqz2(gauss2);
    title('Gaussian Filter \sigma=1');
Gaussian Filter in Space and Freq.

\( \sigma = 1 \)  
\( \sigma = 2 \)

\( \beta = 0.16 \)  
\( \beta = 0.08 \)
Edge Detection

• What is an edge?
• Difference between edge and line and point
• With high resolution images, even a thin line will occupy multiple rows/columns, may have step edges on both sides
Characterization of Edges

Ideal step edge

Real edge has a slope

First order derivative:
Maximum at edge location

Second order derivative:
Zero crossing at edge location
Edge Detection Based on First Order Derivatives

• Edge

High-pass filtering across edge
Low-pass filtering along edge

What if we don’t know edge direction?
Edge Detection Based on Gradients in Two Orthogonal Directions

- Combine results from directional edge detectors in two orthogonal directions and determine the magnitude and direction of the edge.

\[ \sqrt{g} > T \]

\[ \tan^{-1}\left(\frac{f_x}{f_y}\right) \]

\[ f_x^2 + f_y^2 \]
Directional Edge Detector

- High-pass (or band-pass) in one direction (simulating first order derivative)
- Low pass in the orthogonal direction (smooth noise)
- Prewitt edge detector

\[
H_x = \frac{1}{3} \begin{bmatrix}
-1 & -1 & -1 \\
0 & 0 & 0 \\
1 & 1 & 1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} \begin{bmatrix}
1 & 1 & 1
\end{bmatrix};
H_y = \frac{1}{3} \begin{bmatrix}
-1 & 0 & 1 \\
-1 & 0 & 1 \\
-1 & 0 & 1
\end{bmatrix} = \frac{1}{3} \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

- Sobel edge detector

\[
H_x = \frac{1}{4} \begin{bmatrix}
-1 & -2 & -1 \\
0 & 0 & 0 \\
1 & 2 & 1
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
-1 \\
0 \\
1
\end{bmatrix} \begin{bmatrix}
1 & 2 & 1
\end{bmatrix};
H_y = \frac{1}{4} \begin{bmatrix}
-1 & 0 & 1 \\
-2 & 0 & 2 \\
-1 & 0 & 1
\end{bmatrix} = \frac{1}{4} \begin{bmatrix}
1 \\
2 \\
1
\end{bmatrix}
\]

The sobel filter provides better smoothing along the edge.
Freq. Response of Sobel Filter

Sobel Filter for Horizontal Edges:

\[ H_x = \frac{1}{4} \begin{bmatrix} -1 & -2 & -1 \\ 0 & 0 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \]

Frequency Response (DTFT):

\[ h_x = [-1 \ 0 \ 1] \rightarrow H_x = \left( -e^{j2\pi u} + e^{-j2\pi u} \right) = -2j\sin 2\pi u \]

\[ h_y = \frac{1}{4} [1 \ 2 \ 1] \rightarrow H_y = \frac{1}{4} \left( e^{j2\pi v} + 2 + e^{-j2\pi v} \right) = \frac{1}{2} (1 + \cos 2\pi v) \]

![Graphs of |Hx(u)| and |Hy(v)|](image-url)
Spectrum of the Sobel Filter

\( H_x \)

\( H_y \)

Low pass along the edge, band pass cross the edge
Example of Sobel Edge Detector

Original image  Filtered image by $H_x$  Filtered image by $H_y$
How to set threshold?

• Trial and error
• According to edge magnitude distribution (histogram of edge magnitude image)
  – E.g assuming only 5% pixels should be edge pixels, then the threshold should be the 95% percentile of the edge magnitude
Histogram of gm

T=100  T=50  T=20
Problems of previous approach

- Cannot locate edges precisely
- Ramp edges can lead to many edge pixels detected depending on the threshold $T$
  - $T$ too high: may not detect weak edges
  - $T$ too small: detected edges too think, noisy points falsely detected
- Remedy:
  - Detecting local maximum of $|g|$ in the normal direction of the edge, or try all possible 8 directions in a 3x3 neighbor
  - Only consider pixels with $|g| > T$
DoG Filter for Taking Derivatives of Images

• Apply Gaussian filtering first to smooth the image, STD depends on noise level or desired smoothing effect
• Then take derivative in horizontal and vertical directions
• = Convolve the image with a Derivative of Gaussian (DoG) filter

\[
G(x, y) = e^{-\frac{x^2 + y^2}{2\sigma^2}}
\]

\[
H_x(x, y) = \frac{\partial G}{\partial x} = -\frac{x}{\sigma^2} e^{-\frac{x^2 + y^2}{2\sigma^2}}
\]

\[
H_y(x, y) = \frac{\partial G}{\partial y} = -\frac{y}{\sigma^2} e^{-\frac{x^2 + y^2}{2\sigma^2}}
\]

• Sample the above continuous filter to get digital filter. Hy is rotated version of Hx
DoG Filter Examples

s=1, K=5

<table>
<thead>
<tr>
<th></th>
<th>0.0366</th>
<th>0.1642</th>
<th>0.2707</th>
<th>0.1642</th>
<th>0.0366</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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</tr>
<tr>
<td>-0.0821</td>
<td>-0.3679</td>
<td>-0.6065</td>
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<td>-0.2707</td>
<td>-0.1642</td>
<td>-0.0366</td>
<td></td>
</tr>
</tbody>
</table>

S=1, K=3 (similar to Sobel)

<table>
<thead>
<tr>
<th></th>
<th>0.3679</th>
<th>0.6065</th>
<th>0.3679</th>
</tr>
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<tbody>
<tr>
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<td>0.6065</td>
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</tr>
<tr>
<td>-0.3679</td>
<td>-0.6065</td>
<td>-0.3679</td>
<td></td>
</tr>
</tbody>
</table>

DoG filters are low-pass along the edge, and band-pass in orthogonal direction (across edge)
Edge Detection with Many Directional Filters

- Instead of using two orthogonal directions, can design multiple directional filters
  - 0, 45, 90, 135
- See which one gives the highest response in the normal direction
Characterization of Edges

Ideal step edge

Real edge has a slope

First order derivative:
Maximum at edge location

Second order derivative:
Zero crossing at edge location
Edge Detection Based on Second Order Derivative

- Convolve an image with a filter corresponding to taking second order derivative (e.g. Laplacian or applying LoG filter)
- Locate zero-crossing in the filtered image
Laplacian Operator

\[
\nabla^2_f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}
\]

\[
\frac{\partial f}{\partial x} = f(x+1,y) - f(x,y)
\]

\[
\frac{\partial^2 f}{\partial x^2} = f(x+1,y) - 2f(x,y) + f(x-1,y)
\]

\[
\frac{\partial^2 f}{\partial y^2} = f(x,y+1) - 2f(x,y) + f(x,y-1)
\]

\[
\nabla^2_f = [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)] - 4f(x,y)
\]

\[
\begin{bmatrix}
0 & 1 & 0 \\
1 & -4 & 1 \\
0 & 1 & 0
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 0
\end{bmatrix}
\]
Fourier Transform of Laplacian Operator

- The Laplacian operator are isotropic, can detect changes in all directions.
Laplacian of Gaussian (LoG)

- To suppress noise, smooth the signal using a Gaussian filter first
  - $F(x,y) * G(x,y)$
- Then apply Laplacian filter
  - $F(x,y) * G(x,y) * L(x,y) = F(x,y) * (L(x,y) * G(x,y))$
- Equivalent filter: LoG
  - $H(x,y) = L(x,y) * G(x,y)$
Derivation of LoG Filter

• Continuous form

\[ G(x, y) = e^{-\frac{x^2 + y^2}{2\sigma^2}} \]

\[ \nabla^2 G(x, y) = \frac{\partial^2 G}{\partial^2 x} + \frac{\partial^2 G}{\partial^2 y} = \frac{x^2 + y^2 - 2\sigma^2}{\sigma^4} e^{-\frac{x^2 + y^2}{2\sigma^2}} \]

• Take samples to create filter mask
  – Size of mask KxK, K>=5\(\sigma\), odd
  – Ex: \(\sigma=1\), K=5.
LoG Filter

FIGURE 10.21
(a) Three-dimensional plot of the negative of the LoG. (b) Negative of the LoG displayed as an image. (c) Cross section of (a) showing zero crossings. (d) $5 \times 5$ mask approximation to the shape in (a). The negative of this mask would be used in practice.
Laplacian vs. LoG in Freq. Domain

\[ H = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} \]

\[
\begin{array}{cccccc}
0 & 0 & -1 & 0 & 0 \\
0 & -1 & -2 & -1 & 0 \\
-1 & -2 & 16 & -2 & -1 \\
0 & -1 & -2 & -1 & 0 \\
0 & 0 & -1 & 0 & 0 \\
\end{array}
\]
Note that each strong edge in the original image corresponds to a thin stripe with high intensity in one side and low intensity in the other side.
How to detect zero crossing?

• For each pixel that has low filtered value, check a 3x3 neighbor, to see whether its two neighbors in opposite direction have opposite sign and their difference exceeds a threshold (Marr-Hildreth edge detection method)
Example

**FIGURE 10.22**
(a) Original image of size 834 × 1114 pixels, with intensity values scaled to the range [0, 1]. (b) Results of Steps 1 and 2 of the Marr-Hildreth algorithm using \( \sigma = 4 \) and \( n = 25 \). (c) Zero crossings of (b) using a threshold of 0 (note the closed-loop edges). (d) Zero crossings found using a threshold equal to 4% of the maximum value of the image in (b). Note the thin edges.

From: [Gonzalez]
Pros and Cons

• Can locate edges more accurately
• Can detect edges in various direction
• But more prone to noise
• Remedy:
  – Need to select the smoothing parameter $\sigma$ properly
Summary of Edge Detection Method

• First order gradient based:
  – Using edge detectors in two orthogonal directions
    • For each direction: low-pass along edge, band-pass across edge
    • Sobel, DoG filters
  – Using edge detectors in multiple (>2) directions
  – Use threshold or detect local maximum across the edge direction

• Second order gradient based
  – Laplacian is noise-prone, LoG is better
  – Detect zero crossing
  – Isotropic
More on Edge Detection

• Methods discussed so far generally cannot yield connected thin edge maps
• Need sophisticated post processing
  – Thinning
  – Connecting broken lines
• Noise can lead to many false edge points
• Even with many years of research, no perfect edge detectors exist!
  – Canny edge detector: Gaussian smoothing along edges, high pass in each possible edge direction
• For more on edge detection, See Gonzalez Sec. 10.1, 10.2
Results using MATLAB “edge( )” function

Sobel, T=0.14

LOG, T=0.0051

canny, T=[0.0313, 0.0781]

Sobel, T=0.1

LOG, T=0.01

canny, T=[0.1, 0.15]
Line Detection Using Hough Transform

- First do edge detection. Then detect edge pixels falling on a line through a “voting” mechanism

- Pixels \((x,y)\) in a line are related by
  - \(y = ax + b\) or \(\cos \theta x + \sin \theta y = l\)
  - Using a counter for all possible pairs of \((\theta, l)\): Each edgel contributes one count to all possible \((\theta, l)\). \(\Rightarrow H(\theta, l)\): represents the count for the possible line \((\theta, l)\)
  - If there is a line with actual parameter \((\theta^*, l^*)\), \(H(\theta^*, l^*)\) will have a large value!
  - Line detection \(\Rightarrow\) peak detection in \(H(\theta, l)\)

- Generalizable to detecting any shape with analytical parameterization: circles, ovals, etc.

- Optional (Not required, but good to know!)
Image Sharpening (Deblurring)

- **Sharpening**: to enhance line structures or other details in an image.
- Enhanced image = original image + scaled version of the line structures and edges in the image.
- Line structures and edges can be obtained by applying a high pass filter on the image.
- In frequency domain, the filter has the “high-emphasis” character:
  - Design a high-emphasis filter directly.
  - Construct such filter from low pass or high pass filter.
Designing Sharpening Filter Using High Pass Filters

• The desired image is the original plus an appropriately scaled high-passed image
• Sharpening filter

\[ f_s = f + \lambda f_h \]

\[ h_s(m,n) = \delta(m,n) + \lambda h_h(m,n) \]

All pass High pass
Interpretation in Freq Domain

high emphasis=allpass+highpass

all pass

high pass

$0$
High Emphasis Filter Using Laplacian Operator as Highpass Filter

\[ f(x) \]

\[ g(x) = f(x) * h(x) \]

\[ f_s(x) = f(x) + \lambda g(x) \]

\[
H_h = \frac{1}{4} \begin{bmatrix}
0 & -1 & 0 \\
-1 & 4 & -1 \\
0 & -1 & 0
\end{bmatrix} \quad \Rightarrow \quad H_s = \frac{1}{4} \begin{bmatrix}
0 & -1 & 0 \\
-1 & 8 & -1 \\
0 & -1 & 0
\end{bmatrix} \quad \text{with} \quad \lambda = 1.
\]

\[
H_h = \frac{1}{8} \begin{bmatrix}
-1 & -1 & -1 \\
-1 & 8 & -1 \\
-1 & -1 & -1
\end{bmatrix} \quad \Rightarrow \quad H_s = \frac{1}{8} \begin{bmatrix}
-1 & -1 & -1 \\
-1 & 16 & -1 \\
-1 & -1 & -1
\end{bmatrix} \quad \text{with} \quad \lambda = 1.
\]
Example of Sharpening Using Laplacian Operator

(a) Image of the North Pole of the moon.
(b) Laplacian-filtered image.
(c) Laplacian image scaled for display purposes.
(d) Image enhanced by using Eq. (3.7-5).
(Original image courtesy of NASA.)
Summary (1)

• 2D CSFT and DSFT and DFT
  – Many properties of 1D CTFT and DTFT carry over, but there are a few things unique to 2D
  – Meaning of spatial frequency
  – 2D FT of separable signal = product of 1D FT
  – Rotation in space <-> rotation in frequency plane

• 2D linear convolution = weighted average of neighboring pixels
  – Filter=Point spread function (impulse response in 2D)
  – Any LSI (linear and shift invariant) operation can be represented by 2D convolution
  – DSFT of filter = frequency response = response to complex exponential input

• Computation of convolution:
  – boundary treatment, separable filtering

• Convolution theorem
  – Computation of convolution using DFT

• MATLAB function: conv2( ), freqz2( )

• Python: scipy.signal.freqz( )
Summary (2)

- Noise removal using low-pass filters (averaging, Gaussian).
- Edge detection
  - Based on 1st order derivatives: Sobel, DoG (derivative of Gaussian) + finding local maxima
  - Based on 2nd order derivatives: Laplacian, LoG (Laplacian of Gaussian) + finding zero crossing
- Image sharpening by high emphasis filters
  - Filter = low pass + α* high pass
- Given a filter, you should be able to tell what does it do (smoothing, edge detection, sharpening?) by looking at its coefficients, and also through its frequency response
  - Smoothing: coefficient sum = 1, typically all positive
  - Edge detection / high/band pass: coefficient sum = 0
  - Sharpening: coefficient sum = 1, has negative coefficients
Reading Assignments

• Fourier transform and convolution:
  – [Szeliski2012] Richard Szeliski, Computer Vision: Algorithms and Applications. 2012. Ch. 3. (Sec. 3.4.1, 3.4.2) (Does not separately discuss CSFT, DSFT, DFT, very brief)
  – Optional: Wang et al, Digital video processing and communications. Sec. 2.1, 2.2 (for general multi-dimensional case)

• Image smoothing, edge detection and sharpening
Written Assignment (1)

1. Let

\[ f(x, y) = \sin 2\pi f_0 (x + y), \quad h(x, y) = \frac{\sin(2\pi f_c x) \sin(2\pi f_c y)}{\pi^2 xy} \]

Find the convolved signal \( g(x, y) = f(x, y) \ast h(x, y) \) for the following two cases:

a) \( f_0/2 < f_c < f_0 \); and b) \( f_0 < f_c < 2f_0 \).

Hint: do the filtering in the frequency domain. Explain what happened by sketching the original signal, the filter, the convolution process and the convolved signal in the frequency domain.

2. Repeat the previous problem for

\[ h(x, y) = \begin{cases} f_c^2, & -\frac{1}{2f_c} < \{x, y\} < \frac{1}{2f_c} \\ 0, & \text{otherwise} \end{cases} \]
3. For the three filters given below (assuming the origin is at the center):
   a) is the filter separable?
   b) find their Fourier transforms (2D DSFT);
   c) sketch the magnitudes of the Fourier transforms. You should sketch by hand
      the DTFT as a function of u, when v=0 and when v=1/2; also as a function of v,
      when u=0 or ½. Also please plot the magnitude of DSFT as a function of both u
      and v, using Matlab plotting function.
   d) Also use MATLAB freqz2( ) to compute and plot the magnitude of DSFT and
      compare to your answer.
   e) Comment on the functions of the three filters.

In your calculation, you should make use of the separable property of the filter
whenever appropriate. When the filter is not separable, you may be able to split
the filter into several additive terms such that each term can be calculated more
efficiently.

\[
H_1 = \frac{1}{16} \begin{bmatrix}
  1 & 2 & 1 \\
  2 & 4 & 2 \\
  1 & 2 & 1
\end{bmatrix} \\
H_2 = \begin{bmatrix}
  -1 & -1 & -1 \\
  -1 & 8 & -1 \\
  -1 & -1 & -1
\end{bmatrix} \\
H_3 = \begin{bmatrix}
  0 & -1 & 0 \\
  -1 & 5 & -1 \\
  0 & -1 & 0
\end{bmatrix}
\]